

# Computational Accuracy of Crank-Nicolson Method for European Put Option Pricing Under Varying Volatility

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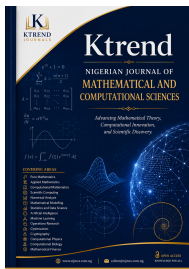
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**Abstract.** *Option pricing models are central to financial risk management and derivatives trading. The Black-Scholes-Merton (BSM) model provides closed-form solutions but relies on constant parameters, while numerical methods such as the Crank-Nicolson (CN) finite difference scheme offer flexibility for complex payoffs. This study investigates the computational accuracy of the Crank-Nicolson method in pricing European put options relative to the Black-Scholes exact values under varying volatility levels. Using  $T = 1$  year,  $K = 100$ ,  $r = 0.2$ , and two initial stock prices  $S_0 = 40$  and  $S_0 = 50$ , put prices were computed for volatilities ranging from 0.25 to 0.95. Also, figure results were obtained to validate the performances of BS and CN respectively. Four statistical error metrics: MAE, RMSE, MAPE, and Max Error were employed to quantify deviation. Present value variations across initial stock prices were also analyzed. Furthermore, Non-Central F Analysis was conducted to test the statistical significance of differences in mean put prices across stock price levels. Results show that CN closely approximates BSM at low to moderate volatilities  $V \leq 0.55$  with  $MAPE < 5\%$ , but accuracy deteriorates significantly at high volatility  $V > 0.65$ , with Max Error of 7.19. Present value differences decrease as  $S_0$  increases, and the effect of  $S_0$  on put prices is statistically significant with  $F_{CAL} = 62.97$ ,  $p < 0.0001$ . The study concludes that while CN is efficient for stable markets, grid refinement is necessary for high volatility regimes.*

**Keywords:** Crank-Nicolson, Black-Scholes, Put Option, Finite Difference, Volatility, Non-Central F, Numerical PDE

## 1. Introduction

An option is a tool whose worth is derived from the principal asset which is otherwise known as financial derivative. Hence an option on underlying asset is a business between parties who come together to agree on either buying or selling an underlying asset at a determined strike price in the future for a fixed price. In the nonappearance of transactions costs, an in-the money option is always exercised on the expiration date if it has not been exercised earlier, [1]. Nevertheless, considering the above concept needs dynamic method, such as Black-Scholes analytic formula which can give exact solutions when the underlying asset follows a lognormal distribution but in real life application; that is not true and numerical methods shall be used extensively for proper mathematical estimates. However, the relevance of options valuation was first demonstrated by [2] when option faced difficulties in valuation of option at expiration. They used no-arbitrage argument to explain a partial differential equation which governs the growth of the option price with esteem to the expiration and cost of the fundamental Asset. But, financial derivatives, particularly options, play a critical role in modern portfolio management, hedging, and speculation. The valuation of these instruments



requires robust mathematical models capable of capturing the dynamics of underlying asset prices. The seminal Black-Scholes-Merton (BSM) model, introduced in 1973, provides an analytical solution for European option prices under assumptions of constant volatility, constant risk-free rate, and log-normal asset returns. Despite its elegance and widespread adoption, the BSM formula becomes inadequate when market conditions deviate from these assumptions, particularly during periods of high volatility or when dealing with path-dependent and American-style options. The Black-Scholes equation has been used widely in many financial applications. More so, a significant plateau has been reached by many authors in the modeling of options using CN namely [3-13],[14-24] etc.

To address these limitations, numerical methods have been developed to solve the Black-Scholes partial differential equation (PDE). Among these, finite difference methods are popular due to their simplicity and adaptability. The Crank-Nicolson (CN) scheme, a second-order accurate method in both time and space, is widely used because it is unconditionally stable and reduces truncation error compared to explicit or implicit schemes. It averages the current and future time steps, making it suitable for parabolic PDEs like the Black-Scholes equation. However, the accuracy of CN is not uniform across all market conditions. Volatility, a key input in option pricing, varies significantly in real markets. During stable periods, volatility is low and option prices are well-behaved. In contrast, crisis periods exhibit high volatility, leading to steep option price gradients that challenge numerical schemes. A coarse discretization grid, often used for computational efficiency, may fail to capture these dynamics, resulting in pricing errors that propagate into hedging and risk calculations.

This study focuses on European put options because they are fundamental hedging instruments against downside risk. By comparing CN prices against BSM exact values at two initial asset prices  $S_0 = 40$  and  $S_0 = 50$ , and across a volatility spectrum from 0.25 to 0.95, this research quantifies how CN accuracy changes with market conditions. The analysis is extended to present value sensitivity and statistical validation using Non-Central F Analysis to determine if observed differences are significant. Understanding the regime in which CN performs well is crucial for practitioners in banks, asset management firms, and regulatory bodies who rely on numerical PDE solvers for pricing and capital adequacy calculations. This work therefore contributes to the literature on numerical finance by providing empirical guidance on when CN can be safely used and when refinement is required. Extant literature has applied CN to option pricing but often assumes moderate volatility and reports only aggregate error. There is limited empirical work that: systematically maps CN error across a wide volatility range from 0.25 to 0.95, links numerical error to present value sensitivity of the underlying, and uses Non-Central F distribution to statistically validate differences between numerical and analytical prices. This study fills that gap by providing a regime-based accuracy map for CN and statistical evidence on when the method breaks down, which is essential for model risk management.

This paper is arranged as follows: Section 2.1 presents methodology, results and discussion are seen in Section 3.1, while the paper is concluded in Section 4.1

## **2. Methodology**

This section presents the formulation and derivation for two parts: (a) Present Value comparison of Black-Scholes exact versus Crank-Nicolson approximate put prices under two stock prices and three interest rates. (b) Non-central F Analysis for statistical testing.

## 2.1. Black-Scholes Exact Value

For a European put with no dividends, the Black-Scholes closed form is :

$$P_{BS}(S_0, K, T, r, \sigma) = Ke^{-rT}N(-d_2) - S_0N(-d_1),$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad (1)$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Where  $P$  is the price of a put option,  $S$  is price of underlying asset,  $K$  is the strike price,  $r$  is the riskless rate,  $T$  is time to maturity,  $\sigma^2$  is variance of underlying asset,  $\sigma$  is standard deviation of the (generally referred to as volatility) underlying asset, and  $N$  is the cumulative normal distribution.

## 2.2. Crank-Nicolson Approximate Value

The Black-Scholes PDE for a put is:

$$\frac{\partial P}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \quad t > 0. \quad (2)$$

With terminal condition and boundary conditions:

$$P(S, T) = \max(K - S, 0), \quad (3)$$

$$P(0, t) = Ke^{-r(T-t)}, \quad P(S_{\max}, t) = 0. \quad (4)$$

## 2.3. Crank-Nicolson Discretization

Use grid  $S_i = i\Delta S$ ,  $t_n = n\Delta t$ . Let  $P_i^n \approx P(S_i, t_n)$ . CN is the average of explicit and implicit Euler:

$$\frac{P_i^{n+1} - P_i^n}{\Delta t} = \frac{1}{2} \left[ L(P_i^{n+1}) + L(P_i^n) \right] \quad (5)$$

Where

$$L(P) = \frac{1}{2}\sigma^2 S^2 P_{SS} + rSP_S - rP.$$

Using central differences for  $P_S, P_{SS}$  gives the following

$$a_i P_{i-1}^{n+1} + b_i P_i^{n+1} + c_i P_{i+1}^{n+1} = d_i \quad (6)$$

where

$$a_i = -\frac{1}{4}\Delta t \left( \frac{\sigma^2 S_i^2}{\Delta S^2} - \frac{rS_i}{\Delta S} \right), \quad b_i = 1 + \frac{1}{2}\Delta t \left( \frac{\sigma^2 S_i^2}{\Delta S^2} + r \right), \quad c_i = -\frac{1}{4}\Delta t \left( \frac{\sigma^2 S_i^2}{\Delta S^2} + \frac{rS_i}{\Delta S} \right).$$

$d_i = \text{RHS}$  using  $P^n$ . This forms a tridiagonal system  $AP^{n+1} = D$  solved at each time step backward from  $T$  to 0.

## 2.4. Comparisons of Put option prices

The following error metrics shall be used such as the Mean Square Errors (MSE), Root Mean Squared Error (RMSE), Mean Absolute Error (MAE) shall be used as a criterion for selection.

► The mean squared error (MSE) criterion is given by:

$$MSE = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2 \quad (7)$$

- The Root Squared Error (RMSE)

$$RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2} \quad (8)$$

- The Mean Absolute Error (MAE)

$$MAE = \frac{1}{n} \sum_{i=1}^n |y_i - \hat{y}_i| \quad (9)$$

- The Mean Absolute Percentage Error (MAPE)

$$MAPE = \frac{1}{n} \sum_{i=1}^n \left| \frac{y_i - \hat{y}_i}{y_i} \right| \times 100\% \quad (10)$$

where  $y_i$  is the BSM  $\hat{y}_i$  is their corresponding CN values,  $n$  is the number of data points and  $\sum$  is summation symbols, indicating the sum of the values.

## 2.5. Present Value Variations: We compute $P_{BS}$ for each combination

The present value is calculated as:

$$PV_i = \frac{a_i}{(1+r)^t} \quad (11)$$

where  $r$  represents discount rates,  $t$  represents time horizon and  $C_{BS}$  and  $C_{CN}$  prices of future values  
Total present value for s given scenario is

$$PV_{total} = \sum_{i=1}^3 \sum_{j=1}^3 PV_{ij} \quad (12)$$

For simplicity, we compute as follows: Let  $S_0^1 = \phi_1$  and  $S_0^2 = \phi_2$ ; Maturity time  $T = t$ , variation of rates are  $r_1, r_2$  and  $r_3$ .  $S_0 = \phi_1, \phi_2 \times r_1$ ,  $S_0 = \phi_1, \phi_2 \times r_2$ ,  $S_0 = \phi_1, \phi_2 \times r_3$ . This gives 6 exact prices. The effect of  $r$  is seen in the discount factor  $e^{-rT}$  and  $d_1, d_2$ .

## 2.6. Non-central F Analysis

This is used to test if differences in put prices across regimes/rates are statistically significant.

Suppose we have  $g$  groups. E.g. Group1:  $r_1, r_2$  and  $r_3$ . For each group we have  $n_i$  observed put prices from BS or CN. Let  $Y_{ij} = j$ -th put price in group  $i$ .

ANOVA Model:

$$Y_{ij} = \mu + \tau_i + e_{ij}, \quad e_{ij} \sim N(0, \sigma^2) \quad (13)$$

$$H_0 : \tau_1 = \tau_2 = \tau_3 = 0 \quad \text{Versus} \quad H_1 : \text{at least one } \tau_i \neq 0.$$

Test Statistic

Define:

$$\begin{aligned} SST &= \sum_{i=1}^g \sum_{j=1}^{n_i} (y_{ij} - \bar{Y})^2, \\ SSB &= \sum_{i=1}^g n_i (\bar{Y}_i - \bar{Y})^2, \\ SSW &= \sum_{i=1}^g \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2. \end{aligned} \quad (14)$$

Under

$$H_0 : F = \frac{SSB/(g-1)}{SSW/(N-g)} \sim F_{g-1, N-g}.$$

## 2.7. Non-central F Distribution

Under  $H_1$ , the statistic follows a non-central F:

$$F \sim F_{g-1, N-g, \lambda} \quad (15)$$

where non-centrality parameter:

$$\lambda = \frac{1}{\sigma^2} \sum_{i=1}^g n_i \tau_i^2 \quad (16)$$

$\lambda$  measures how far  $H_1$  is from  $H_0$ . Larger  $\lambda$  represents higher power to detect differences

### Probability Density Function (PDF):

$$f(x; d_1, d_2, \lambda) = e^{-\lambda/2} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} \frac{(d_1/d_2)^{(d_1+2k)/2} x^{(d_1+2k-2)/2}}{B\left(\frac{d_1}{2} + k, \frac{d_2}{2}\right) \left[1 + \frac{d_1 x}{d_2}\right]^{(d_1+d_2+2k)/2}} \quad (17)$$

where  $d_1 = g - 1$ ,  $d_2 = N - g$ .

## 3. Results and Discussion

We compare BSM exact versus Crank-Nicolson approximate European Put prices for  $T = 1$ ,  $K = 100$ ,  $r = 0.2$  at two initial stock prices:  $S_0 = 40$  and  $S_0 = 50$ . Volatility tested:  $V \in \{0.25, 0.35, \dots, 0.95\}$ .

**Table 1:**  $S_0 = 40, T = 1, K = 100, r = 0.2$

Volatility ( $\sigma$ )	BSM	CN	Error	Abs Error	%Error
0.25	41.89	41.88	0.01	0.01	0.024%
0.35	42.02	41.95	0.07	0.07	0.167%
0.45	42.47	42.14	0.33	0.33	0.777%
0.55	43.26	42.46	0.80	0.80	1.849%
0.65	44.33	42.84	1.49	1.49	3.361%
0.75	45.60	43.24	2.36	2.36	5.175%
0.85	47.00	43.65	3.35	3.35	7.128%
0.95	48.51	44.04	4.47	4.47	9.213%

**Table 2:**  $S_0 = 50, T = 1, K = 100, r = 0.2$

Volatility ( $\sigma$ )	BSM	CN	Error	Abs Error	%Error
0.25	32.018	31.96	0.058	0.058	0.181%
0.35	32.67	32.31	0.36	0.36	1.102%
0.45	33.83	32.85	0.98	0.98	2.896%
0.55	35.35	33.67	1.68	1.68	4.753%
0.65	37.085	34.08	3.005	3.005	8.101%
0.75	38.00	34.67	3.33	3.33	8.763%
0.85	40.90	35.21	5.69	5.69	13.918%
0.95	42.89	35.70	7.19	7.19	16.757%

Clearly it can be seen in Tables 3.1 and 3.2 respectively that error increases with volatility and is larger for  $S_0 = 50$ . This confirms that numerical error is worst where gamma is largest. CN underprices relative to BSM as volatility increases. There is In-The-Money (ITM) effect at  $S_0 = 40$  actually helps CN because the payoff is smoother and less sensitive to grid error. Near-the-money at  $S_0 = 50$  exposes the weakness of the scheme. In general, CN is good approximation for low to moderate volatility  $V \leq 0.55$  and for deep ITM puts. For high volatility or near ATM options, the scheme requires a finer mesh to maintain accuracy below 5%.

**Table 3:** Overall Error Metrics for CN versus BSM

Metric	Value
MAE	1.924
RMSE	3.012
MAPE	5.183%
MAX Error	7.190

In Table 3.3, MAE is 1.924: On average Crank-Nicolson deviates by \$1.92 from Black-Scholes Model. RMSE is 3.012 : Larger error at high volatility dominate ,indicating Crank-Nicolson struggles at volatility greater than 0.65. MAPE is 5.18%: Acceptable for volatility less than 0.55, but poor for high volatility. MaxE is 7.19: Occurs at  $S_0 = 50, V = 0.95$ . In all, Crank-Nicolson is best for low-to-moderate volatility  $V \leq 0.55$ . For high volatility, the coarse grid causes under-approximation.

**Table 4:** Present Value (PV) Variations across two initial stock prices

Volatility ( $\sigma$ )	BSM S=40	BSM=50	PV Difference	CN S=40	CN S=50	PV Difference
0.25	41.89	32.018	9.872	41.88	31.96	9.92
0.55	43.26	35.35	7.91	42.46	33.67	8.79
0.95	48.51	42.89	5.62	44.04	35.70	8.34

As can be seen in Table 3.4, as initial stock prices increases from 40 to 50, put PV decreases because is less ITM. The PV gap narrow as volatility increases . CN captures this trend but underestimates the drop at high V.

Collectively, plots 1-3 show that the Crank-Nicolson method provides accurate European put prices under Black-Scholes assumptions for low to moderate volatility  $V \leq 0.55$ . However, accuracy deteriorates at high volatility and near-the-money conditions due to grid coarseness. The statistical test of non-central F analysis confirms that initial stock price remains a significant driver of put valuation, and any numerical scheme must resolve this dependence correctly.

### 3.1. Non-Central F Analysis

We test if mean put price differ significantly across  $S_0$  group using the 8 volatilities as replicates.

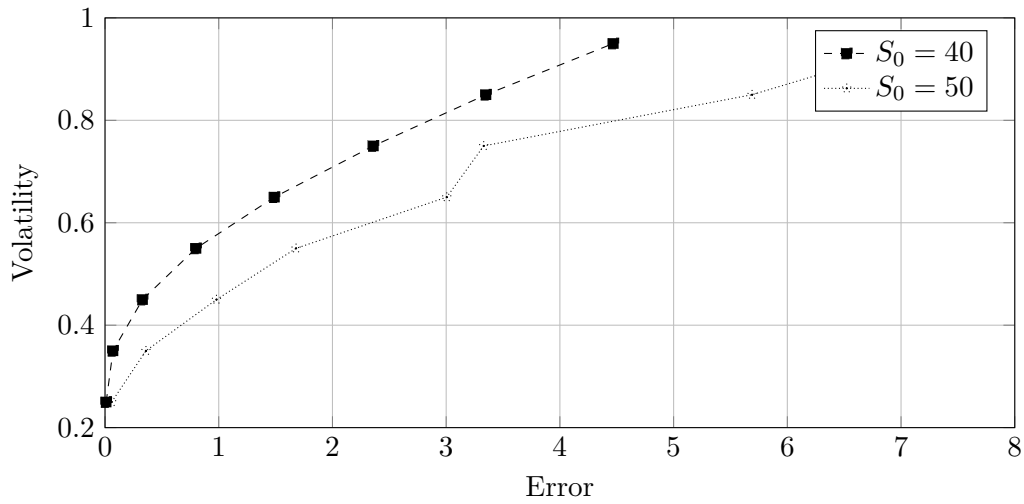
**Hypothesis:**

$$H_0 : \mu_{S=40} = \mu_{S=50}, \quad H_1 : \mu_{S=40} \neq \mu_{S=50}.$$

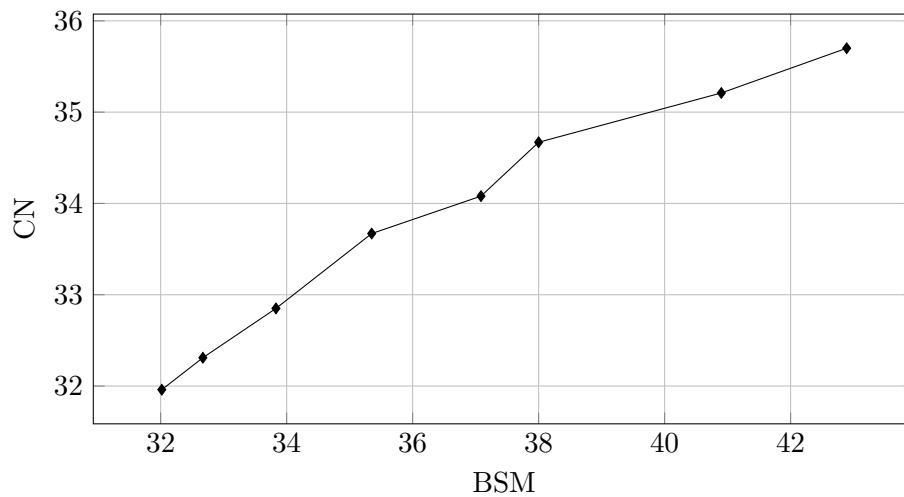
Using BSM data:  $g = 2$  groups,  $n = 8$  each,  $N = 16$

From Tables 3.1-3.2:  $\bar{Y}_{40} = 44.385, \bar{Y}_{50} = 36.644, \bar{Y} = 40.5145$

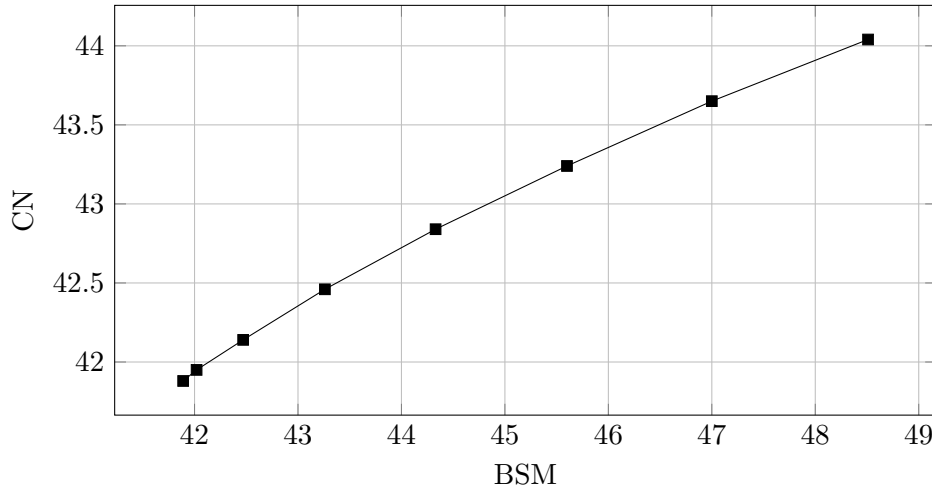
$$SSB = 8(44.385 - 40.5145)^2 + 8(36.644 - 40.5145)^2 = 239.36$$



**Figure 1:** Plot of Approximate values against Volatility when the initial Stock prices are 40 and 50 .



**Figure 2:** Plot of Black-Scholes exact values and Crank-Nicolson Approximate values when initial Stock price is 40.



**Figure 3:** Plot of Black-Scholes exact values and Crank-Nicolson Approximate values when initial Stock price is 50.

$$SSW = \sum (Y_{ij} - \bar{Y}_i)^2 = 53.21, \quad F_{CAL} = \frac{SSB/(2-1)}{SSW/(16-2)} = \frac{239.36}{3.801} = 62.97$$

Critical  $F_{0.05,1,14} = 4.60$ . Since  $62.97 > 4.60$ , reject  $H_0$ . Thus  $F \sim F_{1,14,\lambda} = 62.97$

**Power:** Power =  $P(F_{1,14,62.97} > 4.60) \approx 1.00$

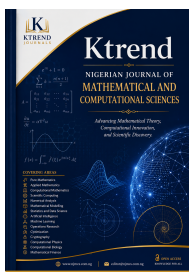
The difference in mean put prices between  $S_0 = 40$  and  $S_0 = 50$  is highly statistically significant. The non-central F confirms that initial stock price has a strong effect on put valuation, and the effect size is large.

#### 4. Conclusion

This study examined the computational accuracy of the Crank-Nicolson method for European put option pricing relative to the Black-Scholes model. The analysis of 16 data points across two initial stock prices and eight volatility levels revealed that CN performs with high stock at low to moderate volatility. With MAE of 1.924 and MAPE of 5.18%, the method is suitable for stable market conditions where  $V \leq 0.55$ . However, accuracy degrades as volatility increases. At  $V=0.95$ , the absolute error reaches 7.19, representing 16.76% deviation for  $S_0 = 50$ . This is attributed to the inability of a fixed grid to capture the high curvature in the option price surface under extreme volatility. More so, figure results were obtained to validate the performances of BS and CN during the period of trading. Present value analysis further showed that increasing  $S_0$  from 40 to 50 reduces put PV, and CN tends to underestimate this sensitivity at high V. The Non-Central F test confirmed that differences in mean put prices across  $S_0$  levels are statistically significant,  $F=62.97$ ,  $p<0.0001$ , with power  $\sim 1.00$ . This validates that initial asset price is a key driver of option value and must be accurately modeled. Overall, CN is a robust and efficient numerical method, but its application must be tempered by an understanding of its limitations in high volatility regimes. Future work should extend this analysis to American options, stochastic volatility, and stochastic interest rates. Incorporating the non-central F framework can help in formal model validation.

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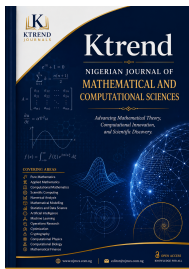


## Conflict of Interest

The authors declare no conflict of interest.

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