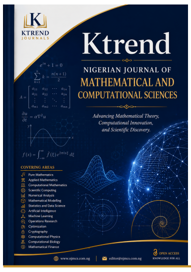

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Extended Green Relations and Rank Pair Enumeration in Finite Rhotrix Semigroups

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Abstract. *Let q be a prime power and let $T_m(q)$ be the monoid of order-$(2m-1)$ Rhotrix arrays whose row–column multiplication corresponds to componentwise multiplication in $M_m(\mathbb{F}_q) \times M_{m-1}(\mathbb{F}_q)$. The extended Green relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are characterized by the row and column spaces of both matrix components. Regularity implies $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$, while $\mathcal{D}^* = \mathcal{D} = \mathcal{J}$ is indexed by the pair of component ranks. For each rank pair, closed formulas are derived for the sizes and numbers of $\mathcal{L}^*$-, $\mathcal{R}^*$-, and $\mathcal{H}^*$-classes, the number of idempotents, and the order of every maximal subgroup. The principal ideal order, all two-sided ideals, and a rank-generating polynomial are determined. An exhaustive computation for $T_2(2)$ confirms the formulas and the translation-kernel definitions. These results provide an explicit finite-field structure theory for the specified row–column Rhotrix monoid.*

Keywords: Rhotrix semigroup; extended Green relations; matrix monoid; finite field; rank pair.

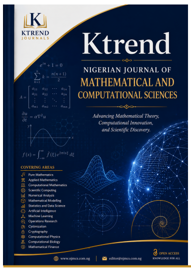
Mathematics Subject Classification (2020): 20M12; 20M20; 15B33.

1. Introduction

The study of rhotrices began with Ajibade's rhomboidal arrays and heart-oriented multiplication, following the matrix-tertion and matrix-noitret constructions of Atanassov and Shannon [1, 4]. Sani introduced a different, row–column multiplication and subsequently a coupled-matrix interpretation [22–24]. The two products need not have the same algebraic properties. Mohammed and collaborators studied semigroup constructions based on row–column multiplication, while later work examined type A semigroups and their representations [13, 16, 17]. The review by Mohammed and Balarabe describes the division between commutative heart-based and noncommutative row–column rhotrix theory [28].

Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$ encode divisibility through principal left, right and two-sided ideals [5, 7–9]. The extended relations $\mathcal{L}^*, \mathcal{R}^*$ instead compare the kernels of right and left translations on a semigroup. This extension is especially informative in nonregular semigroups; it underlies the theory of abundant and related semigroups [6, 11, 15]. For a regular semigroup, however, each starred one-sided relation agrees with its ordinary counterpart. Therefore a finite-field pair of full matrix monoids, although it has a rich Green stratification, cannot exhibit a strict distinction between its ordinary and starred Green classes.

The coupled-matrix viewpoint raises three precise questions. First, what information about the two components detects each extended Green relation? Second, how many elements, one-sided classes, groups and idempotents occur at each pair of ranks? Third, what is the complete partial order of two-sided ideals? Answering them requires specifying the product and whether the two component matrices are independent. An embedding alone is insufficient: a proper subsemigroup of



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a product can have different ideals and starred relations. The object is therefore defined through the unrestricted pair construction in Section 2. Its isomorphism with the full product is verified, and every result is stated within that scope.

The rank-pair classification and its enumeration are obtained in Sections 3 and 4. Section 5 develops the ideal lattice, rank polynomial and maximal subgroup structure. Section 6 checks the smallest nontrivial case directly, including the defining translation-kernel condition. The discussion identifies the boundary of the results for constrained variants and explains why Gram matrices and transpose-based inverses cannot replace the kernel definition over an arbitrary finite field. Related work on rhotrix groups, ideals over rings and semigroup rank provides context for this distinction [27, 29–33]. The formulas here are consequences of established matrix-monoid methods applied systematically to the specified Rhotrix model; journal novelty must be judged against the specialized rhotrix literature.

2. Pair model and definitions

Let q be a prime power, $m \geq 2$, $k = m-1$, and $F = \mathbb{F}_q$. Define the order-$(2m-1)$ Rhotrix semigroup by

$$T_m(q) = \{\langle A, B\rangle : A \in M_m(F), B \in M_k(F)\}, \quad \langle A, B\rangle\langle C, D\rangle = \langle AC, BD\rangle. \tag{1}$$

The definition assumes freely and uniquely specified major and minor entries; a geometrically constrained variant need not satisfy (1). It is a monoid isomorphic to $M_m(F) \times M_k(F)$. For a monoid S, define $a\mathcal{R}^*b$ if $xa = ya \Leftrightarrow xb = yb$ for every $x, y \in S$; define $a\mathcal{L}^*b$ dually with right multiplication. Put $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$ and $\mathcal{D}^* = \mathcal{R}^* \vee \mathcal{L}^*$. The notation $\mathcal{J}^*$ is avoided because its definition varies across sources. Row and column spaces have their usual linear-algebra meanings. For the standard odd-order diamond arrangement, the count of free positions is $m^2 + (m-1)^2 = 2m^2 - 2m + 1$, the number of entries in an odd-order diamond of order $2m - 1$. The correspondence is a bijection because the two arrays of matrix entries occupy complementary positions. Applying the row–column product separately to the two matrices gives (1), not merely an injective homomorphism into a larger matrix monoid. This surjectivity is essential to all class and ideal formulas below. The identity is $\langle I_m, I_k\rangle$ and the zero is $\langle 0_m, 0_k\rangle$.

Classical notation is fixed as follows: $x\mathcal{R}y$ if $xT_m(q) = yT_m(q)$, $x\mathcal{L}y$ if $T_m(q)x = T_m(q)y$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$, and $x\mathcal{J}y$ if their principal two-sided ideals agree. A rank pair is $\rho(\langle A, B\rangle) = (\operatorname{rank} A, \operatorname{rank} B)$. No involution is required to define the starred relations. This point is relevant because "star" in $\mathcal{L}^*$ and $\mathcal{R}^*$ is unrelated to a matrix adjoint.

Proposition 2.1. *Every element of $T_m(q)$ is regular and its units form $\mathrm{GL}_m(q) \times \mathrm{GL}_k(q)$.*

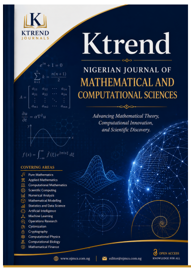
Proof. For $A = P \operatorname{diag}(I_r, 0)Q$ with P, Q invertible, $A' = Q^{-1} \operatorname{diag}(I_r, 0)P^{-1}$ satisfies $AA'A = A$. Apply this construction independently to both components. Units in a direct product are componentwise units. □

3. Relations and ideal order

The following proposition records the linear-algebra mechanism behind the starred relations. It avoids reliance on an involution and makes the direction of the row/column criteria explicit.

Proposition 3.1. *For $A, C \in M_n(F)$, the kernels of the right-translation maps $X \mapsto XA$ and $X \mapsto XC$ agree if and only if $\operatorname{Col} A = \operatorname{Col} C$. Dually, the kernels of left-translation maps $X \mapsto AX$ and $X \mapsto CX$ agree if and only if $\operatorname{Row} A = \operatorname{Row} C$.*

Proof. The equation $XA = YA$ means that every row of $X - Y$, regarded as a linear functional on F^n, vanishes on $\operatorname{Col} A$. Consequently its solution pairs depend exactly on the annihilator $(\operatorname{Col} A)^{\perp}$.



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Equality of these annihilators gives equality of column spaces in finite dimension. Applying the same argument to transposes gives the left-translation statement. The use of a transpose here is only a convenient way to describe row spaces; it is not the definition of a starred relation. □

For pairs, choose X, Y with equal second component to isolate the major factor and equal first component to isolate the minor factor. Thus equality of translation kernels in (1) is precisely equality of each relevant component kernel.

Theorem 3.2. *For $x = \langle A, B\rangle$ and $y = \langle C, D\rangle$,*

$$\begin{aligned} x\mathcal{R}^* y &\iff \operatorname{Col} A = \operatorname{Col} C \ \textit{and}\ \operatorname{Col} B = \operatorname{Col} D,\\ x\mathcal{L}^* y &\iff \operatorname{Row} A = \operatorname{Row} C \ \textit{and}\ \operatorname{Row} B = \operatorname{Row} D,\\ x\mathcal{D}^* y &\iff (\operatorname{rank} A, \operatorname{rank} B) = (\operatorname{rank} C, \operatorname{rank} D). \end{aligned}$$

Furthermore $\mathcal{R}^* = \mathcal{R}$, $\mathcal{L}^* = \mathcal{L}$, $\mathcal{H}^* = \mathcal{H}$ and $\mathcal{D}^* = \mathcal{D} = \mathcal{J}$.

Proof. In a regular monoid, $\mathcal{R}^* = \mathcal{R}$: if $a = axa$ and $a\mathcal{R}^*b$, then $1a = (ax)a$ implies $b = (ax)b \in aS$; the reverse argument gives $a \in bS$. Conversely, equality of principal right ideals implies equality of kernels of right multiplication, by writing each element as a right multiple of the other. The left statement is dual. In $M_n(F)$, AS comprises exactly matrices whose columns lie in $\operatorname{Col} A$, while SA is described by row spaces. Direct products yield the first two equivalences. For fixed rank r, any specified r-dimensional row and column spaces admit a rank-r matrix with those spaces, so $\mathcal{R}$ and $\mathcal{L}$ connect precisely the common-rank matrices. The two components connect independently. Finally SAS in a full matrix monoid consists of matrices of rank at most $\operatorname{rank} A$, which proves $\mathcal{D} = \mathcal{J}$. □

Proposition 3.3. *For $x = \langle A, B\rangle$ of rank pair (r, s),*

$$T_m(q)xT_m(q) = \{\langle C, D\rangle : \operatorname{rank} C \le r,\ \operatorname{rank} D \le s\}. \tag{2}$$

Thus principal ideals form the product order on $\{0, \dots, m\} \times \{0, \dots, k\}$. Every two-sided ideal is a downward-closed union of rank-pair strata.

Proof. The rank factorization criterion $C = XAY \iff \operatorname{rank} C \le \operatorname{rank} A$ applies in each component independently. □

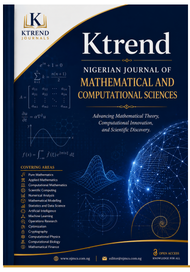
A Gram-matrix criterion cannot replace the translation-kernel definition. It fails even over F_2: $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ have equal column spaces and hence are $\mathcal{R}^*$-related, but $A^{\mathsf{T}}A \ne C^{\mathsf{T}}C$. Equal minor components give a paired counterexample. A Moore–Penrose inverse associated with transpose does not exist for every matrix over every field; ordinary inner inverses above suffice [18, 20].

4. Finite-field counts

Put $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_q = \prod_{i=0}^{r-1}(q^{n-i} - 1)/(q^{r-i} - 1)$ and $g_r(q) = \prod_{i=0}^{r-1}(q^r - q^i)$, with empty products one. Write $N_{n,r} = \left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_q^2 g_r(q)$ [12, 25].

Theorem 4.1. *There are $(m+1)(k+1)$ $\mathcal{D}^*$-classes. The rank-(r, s) class contains $N_{m,r}N_{k,s}$ elements, $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_q \left[\begin{smallmatrix} k \\ s \end{smallmatrix}\right]_q$ right classes and the same number of left classes. Each of its nonempty $\mathcal{H}^*$-classes has $g_r(q)g_s(q)$ elements. It contains*

$$\begin{bmatrix} m \\ r \end{bmatrix}_q q^{r(m-r)} \begin{bmatrix} k \\ s \end{bmatrix}_q q^{s(k-s)}$$



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idempotents. The whole monoid has $q^{m^2+k^2}$ *elements. Its principal ideal at* (r,s) *has*

$$\left(\sum_{i=0}^{r} N_{m,i}\right)\left(\sum_{j=0}^{s} N_{k,j}\right)$$

elements.

Proof. There are $\begin{bmatrix} n \\ r \end{bmatrix}_q$ choices for each rank-r image and row space. Given both, a matrix induces an isomorphism from an r-dimensional quotient to its image, with $g_r(q)$ choices. Theorem 3.2 gives the Green class counts. A rank-r idempotent projects onto a chosen r-space along a complement; the $\begin{bmatrix} n \\ r \end{bmatrix}_q$ images each admit $q^{r(n-r)}$ complements. Multiply independent component counts. The remaining totals follow from (1) and (2). □

5. Ideal lattice, rank polynomial and local groups

Theorem 5.1. *The assignment* $I \mapsto \{(r,s) :$ *a rank-*(r,s) *element lies in* $I\}$ *is a bijection between two-sided ideals of* $T_m(q)$ *and nonempty lower sets of the grid* $[0,m]\times[0,k]$*. The total number of nonempty two-sided ideals is*

$$\binom{m+k+2}{m+1} - 1 = \binom{2m+1}{m+1} - 1. \tag{3}$$

Exactly $(m+1)(k+1)$ *of these ideals are principal. The unique minimal ideal is the zero singleton.*

Proof. An ideal is a union of $\mathcal{J}$-classes, and (2) shows that membership at (r,s) forces membership at every coordinatewise smaller pair. Conversely a nonempty lower set gives a union closed under multiplication on both sides because matrix ranks cannot increase under multiplication. Lower sets of a rectangle with $m+1$ rows and $k+1$ columns correspond to monotone lattice paths along its boundary, of which there are $\binom{m+k+2}{m+1}$, including the empty lower set. Each grid point generates one distinct principal ideal. The bottom rank pair consists only of zero. □

Define the bivariate rank enumerator $P_m(u,v) = \sum_{x\in T_m(q)} u^{\operatorname{rank} A_x} v^{\operatorname{rank} B_x}$. Independence of components gives the factorization

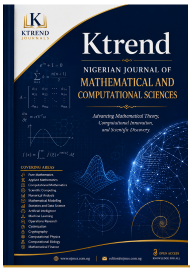
$$P_m(u,v) = \left(\sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q^2 g_r(q) u^r\right)\left(\sum_{s=0}^{k} \begin{bmatrix} k \\ s \end{bmatrix}_q^2 g_s(q) v^s\right). \tag{4}$$

Here and below $\begin{bmatrix} n \\ r \end{bmatrix}_q$ is the Gaussian binomial coefficient defined in Section 4. Evaluating at $(1,1)$ recovers $q^{m^2+k^2}$; truncating the sums at (r,s) gives the principal-ideal size. The corresponding idempotent rank polynomial is

$$E_m(u,v) = \left(\sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q q^{r(m-r)} u^r\right)\left(\sum_{s=0}^{k} \begin{bmatrix} k \\ s \end{bmatrix}_q q^{s(k-s)} v^s\right). \tag{5}$$

Proposition 5.2. *Every idempotent* e *of rank pair* (r,s) *lies in a maximal subgroup isomorphic to* $\mathrm{GL}_r(q)\times\mathrm{GL}_s(q)$*; its order is* $g_r(q)g_s(q)$*. At rank pair* (r,s) *the number of group* $\mathcal{H}$*-classes equals the idempotent count in Theorem 4.1. The remaining* $\mathcal{H}$*-classes contain no idempotent and are not groups.*

Proof. Choose bases adapted to the image and kernel of each idempotent. In these bases its two components have block form $\operatorname{diag}(I_r,0)$ and $\operatorname{diag}(I_s,0)$. The $\mathcal{H}$-class at this pair consists precisely of $\operatorname{diag}(G,0)$ and $\operatorname{diag}(H,0)$ for $G\in\mathrm{GL}_r(q)$ and $H\in\mathrm{GL}_s(q)$, with componentwise multiplication. A group $\mathcal{H}$-class has exactly one idempotent, its identity. Apply the idempotent formula in Theorem 4.1. □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6. Complete order-three check

For $m = 2, k = 1, q = 2$, the rank distributions in the two factors are $(1, 9, 6)$ and $(1, 1)$. The six paired strata have sizes $1, 9, 6, 1, 9, 6$, summing to 32. The $(1, 1)$ stratum has nine elements, three classes of each one-sided type, singleton $\mathcal{H}^*$-classes, and six idempotents. The full distribution for the two smallest fields is displayed in Table 1. The row labelled $(1, 1)$ gives a local comparison of one-sided class counts and maximal subgroup orders; the latter are orders only for group $\mathcal{H}$-classes, not for every $\mathcal{H}$-class. For $q = 3$ the $(1, 1)$ stratum has twelve idempotents and sixteen $\mathcal{H}$-classes, each

Table 1: Rank-pair class sizes for the order-three rhotrix semigroup over two finite fields

q	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	total	$\mathcal{R}$ at $(1,1)$	group order
2	1	9	6	1	9	6	32	3	1
3	1	32	48	2	64	96	243	4	4

of size four. Twelve of those classes contain an idempotent and are groups, while four contain none. This distinction illustrates why the size of an $\mathcal{H}$-class does not by itself make it a subgroup. The following Python 3 code enumerates the binary case and checks the kernel definition and regularity without a dependency:

```
from itertools import product
from collections import Counter
F=range(2); M=list(product(F,repeat=4))
def mm(a,b):
 return tuple(sum(a[2*i+t]*b[2*t+j] for t in F)%2
              for i in F for j in F)
def rank(a):
 return 2 if (a[0]*a[3]-a[1]*a[2])%2 else int(any(a))
S=list(product(M,F))
def op(x,y): return (mm(x[0],y[0]),x[1]*y[1])
assert Counter((rank(a),b) for a,b in S)=={
 (0,0):1,(1,0):9,(2,0):6,(0,1):1,(1,1):9,(2,1):6}
assert all(any(op(op(x,y),x)==x for y in S) for x in S)
def rstar(x,y):
 return all((op(u,x)==op(v,x))==(op(u,y)==op(v,y))
            for u in S for v in S)
def ideal(x): return {op(x,u) for u in S}
assert all(rstar(x,y)==(ideal(x)==ideal(y))
           for x in S for y in S)
assert sum(op(x,x)==x for x in S if rank(x[0])==1
           and x[1]==1)==6
print('verified')
```

7. Further structural consequences

The formulas distinguish two different notions of size. A rank-(r, s) $\mathcal{D}$-class has

$$\begin{bmatrix} m \\ r \end{bmatrix}_q^2 \begin{bmatrix} k \\ s \end{bmatrix}_q^2 g_r(q) g_s(q)$$

elements, while its principal ideal also contains every lower-rank stratum. The difference is substantial near the top of the grid. For example, at the identity rank pair (m, k) the $\mathcal{D}$-class is the unit group

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with $g_m(q)g_k(q)$ elements, whereas its principal ideal is the entire monoid of size $q^{m^2+k^2}$. At rank pair $(0,0)$ both reduce to the zero singleton.

An $\mathcal{R}$-class of rank pair (r,s) fixes a pair of image subspaces but permits any row subspaces of matching dimensions. It therefore contains $\begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} k \\ s \end{bmatrix}_q g_r(q)g_s(q)$ elements. The same number belongs to an $\mathcal{L}$-class. Dividing by $g_r(q)g_s(q)$ gives the number of $\mathcal{H}$-classes per one-sided class. These identities offer an independent arithmetic check on Theorem 4.1.

The counts of regular elements and idempotents should not be confused. Every one of the $q^{m^2+k^2}$ elements is regular in the semigroup sense, but only $E_m(1,1)$ is idempotent. For $T_2(2)$, $E_2(1,1) = (1+6+1)(1+1) = 16$. For $T_2(3)$ the corresponding value is $(1+12+1)(1+1) = 28$. By Proposition 5.2, these are also the numbers of group $\mathcal{H}$-classes in the respective monoids.

8. Discussion and conclusion

The pair representation makes the extended Green problem transparent: a right class fixes both images, a left class fixes both row spaces, and a two-sided class fixes both ranks. The finite-field formulas quantify this structural description and expose a grid of principal two-sided ideals. The number of all ideals grows combinatorially as in (3), whereas the number of $\mathcal{D}^*$-classes is only $(m+1)m$. The distinction between a class and its principal factor matters: a product of two elements of a fixed positive rank pair may lose rank, so a $\mathcal{D}$-class is not generally a subsemigroup. Rees matrix theory concerns the corresponding principal factor after adjoining a zero for products leaving the class [5, 21].

The hypotheses cannot be dropped silently. If entries of the two coupled matrices are required to agree or obey other constraints, the resulting subset may fail to contain the inner inverses used in Section 2, and $\mathcal{R}^*$ or $\mathcal{L}^*$ may then be strictly larger than the ordinary relations. Likewise, a rhotrix over a ring or semiring requires a different treatment because rank normal form over a field is no longer available in this form [30]. Future work should specify an actual constrained product and compute its translation kernels directly. The present theorem supplies the baseline against which such departures can be measured.

Statements and provenance

No external dataset was used; the complete verification code is included in Section 6. Funding and conflict-of-interest statements require confirmation from the author. This paper was developed using a coursework seminar attributed to Pitam Friday, with J. U. Udo-Akpan and O. G. Udoaka named as supervisors. Authorship and acknowledgement of the underlying work require the contributors' agreement before submission.

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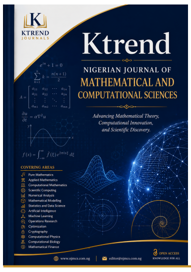

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