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Structural Properties of Suborbital Graphs under Product Actions of Wreath Products

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Abstract. *This paper examines the suborbital graphs generated by the product actions of the wreath products $S_m \wr S_n$, $A_m \wr S_n$, $C_m \wr S_n$, and $D_{2m} \wr S_n$ on the Cartesian power X^n. Using permutation group theory, orbit and stabilizer analysis, combinatorial techniques, and computational verification, the study investigates the classification of suborbits and subdegrees and the structural properties of the resulting graphs. The analysis focuses on the pairing properties of orbitals and graph characteristics, including connectivity, vertex degree, regularity, transitivity, girth, and symmetry. The findings reveal how the orbit structure of each base group, together with coordinate permutations, determines the structure of the associated suborbital graphs. In particular, under suitable conditions, the symmetric and alternating group actions yield adjacency structures governed by Hamming distance, whereas the cyclic and dihedral group actions produce graph structures reflecting their respective permutation actions. The study provides a group-theoretic framework for examining suborbital graphs arising from wreath product actions and contributes to the understanding of the relationship between permutation groups, Cartesian powers, and algebraic graph structures.*

Keywords: wreath products; product action; suborbital graphs; orbital graphs; Cartesian powers; permutation groups; Hamming graphs

Mathematics Subject Classification (2020): 05C25; 20B25; 20B15; 05C12; 05C50

1. Introduction

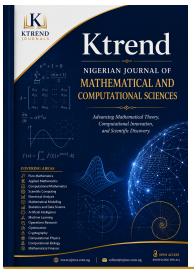
Let $X = \{1, 2, \ldots, m\}$, where $m \geq 3$, and let G be a permutation group acting on X. For an integer $n \geq 2$, consider the wreath product

$$W = G \wr S_n = G^n \rtimes S_n,$$

acting in product action on the Cartesian power

$$\Omega = X^n.$$

The action combines the coordinatewise action of the base group G^n with the natural permutation action of S_n on the coordinates. Thus, the resulting permutation group acts on n-tuples by applying elements of G to the individual coordinates and permuting their positions. Product actions of wreath products form an important class of permutation group actions and are closely connected with the study of Cartesian decompositions and the structure of permutation groups [4, 5, 8, 9].



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A central concept in the study of permutation groups is the orbital structure of a group action. Given a transitive permutation group W acting on Ω, its orbits on $\Omega \times \Omega$ are called orbitals. Each orbital determines a directed orbital graph, commonly referred to as a suborbital graph. The properties of these graphs are closely related to the suborbit structure of the corresponding permutation group. In particular, the relationship between an orbital and its paired orbital determines whether the associated suborbital graph is self-paired and hence whether its adjacency relation defines an undirected graph. The study of suborbital graphs has received considerable attention, particularly in connection with modular groups and their associated permutation actions [1, 2].

The structural properties of orbital graphs also connect permutation group theory with several areas of algebraic and algebraic-combinatorial graph theory. For example, Hamming distance provides a natural description of adjacency in graphs arising from Cartesian powers, while association schemes and distance-regular graphs provide broader frameworks for investigating regularity and distance properties [3, 6, 7, 10]. These connections provide useful foundations for examining the graph structures generated by product actions.

Despite these developments, the construction and comparative analysis of suborbital graphs arising from different classical permutation groups under the same wreath product product action require further investigation. In particular, the effects of changing the base group while retaining the same top group and Cartesian domain need to be examined through the resulting orbital structures and graph properties. This study addresses this issue by considering the natural permutation representations of the symmetric group S_m, alternating group A_m, cyclic group C_m, and dihedral group D_{2m}, each acting on X, with the symmetric group S_n acting on the coordinate positions. The corresponding wreath products are therefore

$$S_m \wr S_n, \qquad A_m \wr S_n, \qquad C_m \wr S_n, \qquad D_{2m} \wr S_n,$$

acting on $\Omega = X^n$.

The main objective of this research is to construct and investigate the suborbital graphs induced by these four product actions and to determine how the algebraic structure of the base group influences their graph-theoretic properties. Specifically, the study examines the orbitals and their paired orbitals, classifies the resulting suborbital graphs according to their pairing properties, and investigates their structural features, including regularity, connectivity, diameter, girth, bipartiteness, and multipartite structure. It also establishes connections between the resulting graphs and familiar graph families, including Hamming-type, Cartesian, Cayley, and complete multipartite graphs, where supported by the corresponding orbital structures.

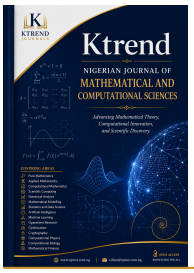
The contribution of this work lies in providing a systematic comparison of the suborbital graph structures generated by symmetric, alternating, cyclic, and dihedral base groups under wreath product actions. By keeping the Cartesian domain and the top group fixed while varying the base permutation group, the study identifies how differences in the underlying permutation actions are reflected in orbital pairing and graph structure. The results therefore contribute to the understanding of the interaction between permutation group actions and the combinatorial properties of their associated suborbital graphs.

2. Results

2.1. Connectivity of Suborbital Graphs

Definition 2.1 (Connected Component)**.** Let $\Gamma = (\Omega, E)$ be an undirected graph. For $\alpha \in \Omega$, define

$$C_\Gamma(\alpha) = \left\{ \beta \in \Omega : \begin{array}{l} \text{there exist } k \geq 0 \text{ and } v_0, \ldots, v_k \in \Omega \\ \text{such that } v_0 = \alpha,\ v_k = \beta, \text{ and} \\ \{v_{i-1}, v_i\} \in E, \quad 1 \leq i \leq k \end{array} \right\}.$$



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The graph Γ is connected if and only if

$$C_\Gamma(\alpha) = \Omega \qquad \text{for every } \alpha \in \Omega.$$

Theorem 2.2 (Connectivity Criterion)**.** *Let $W \leq \mathrm{Sym}(\Omega)$ act transitively on Ω, let $\alpha \in \Omega$, and put*

$$H = W_\alpha = \{w \in W : \alpha^w = \alpha\}.$$

Let $g \in W$ and $\beta = \alpha^g$, where $\beta \neq \alpha$. Let Γ be the undirected orbital graph whose edge relation is the union of the orbital of (α, β) and its paired orbital.

Then

$$\boxed{\Gamma \text{ is connected} \iff \langle H, g\rangle = W.}$$

Proof. Let

$$L = \langle H, g\rangle \leq W.$$

We prove the two implications in (2.2).

Step 1: Determine the edge relation.

Using the right-action convention, the orbital of (α, α^g) is

$$\mathcal{O} = \{(\alpha^w, \alpha^{gw}) : w \in W\}.$$

Its paired orbital is

$$\mathcal{O}^* = \{(\alpha^{gw}, \alpha^w) : w \in W\}.$$

Hence the undirected edge relation is obtained from

$$E = \mathcal{O} \cup \mathcal{O}^*.$$

For every $w \in W$, the pair

$$(\alpha^w, \alpha^{gw})$$

is an arc of $\mathcal{O}$. Since

$$(\alpha^w)^g = \alpha^{gw},$$

the element g describes the transition between the two entries of this representative ordered pair.

Step 2: Show that connectivity implies $\langle H, g\rangle = W$.

Suppose that Γ is connected. Then, for every $\delta \in \Omega$, there exist

$$k \geq 0, \qquad v_0, v_1, \ldots, v_k \in \Omega$$

such that

$$v_0 = \alpha, \qquad v_k = \delta, \qquad \{v_{i-1}, v_i\} \in E \quad (1 \leq i \leq k).$$

By the orbital definition in (2.1)–(2.1), each edge in the path is obtained from a pair in $\mathcal{O}$ or $\mathcal{O}^*$ by an element of W.

Consequently, connectivity says that the orbit of α under the subgroup generated by the point stabilizer and the elements that generate the orbital transitions is all of Ω. In the usual orbital connectivity criterion, this gives

$$\alpha^{\langle H,g\rangle} = \Omega.$$

Since $H \leq L$, equation (2.1) implies that L is transitive on Ω. We now show algebraically that $L = W$.

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Let $w \in W$ be arbitrary. By the transitivity of L, there exists $\ell \in L$ such that

$$\alpha^{\ell} = \alpha^{w}.$$

Applying ℓ^{-1} on the right gives

$$\alpha^{w\ell^{-1}} = \alpha.$$

Therefore,

$$w\ell^{-1} \in W_{\alpha} = H.$$

Because $H \leq L$ and $\ell \in L$, we obtain

$$w = (w\ell^{-1})\ell \in L.$$

Since $w \in W$ was arbitrary,

$$W \leq L.$$

On the other hand, $L = \langle H, g \rangle \leq W$. Thus

$$L = W,$$

or equivalently,

$$\langle H, g \rangle = W.$$

Step 3: Show that $\langle H, g \rangle = W$ implies connectivity.

Conversely, suppose that

$$\langle H, g \rangle = W.$$

Let $\delta \in \Omega$. Since W is transitive, there exists $w \in W$ such that

$$\delta = \alpha^{w}.$$

By (2.1), every element of W, including w, belongs to $\langle H, g \rangle$. Thus w can be expressed as a finite product of elements of H, g, and their inverses:

$$w = h_0 g^{\varepsilon_1} h_1 g^{\varepsilon_2} \cdots g^{\varepsilon_k} h_k,$$

where $h_i \in H$ and $\varepsilon_i \in \{1, -1\}$, with the factors arranged according to the chosen product convention.

The elements of H fix α, and the orbital transitions represented by g and g^{-1} correspond to edges of the orbital graph or its paired orbital. Therefore, the successive factors in (2.1) give a finite walk from α to $\alpha^{w} = \delta$.

Hence

$$\delta \in C_{\Gamma}(\alpha).$$

Since $\delta \in \Omega$ was arbitrary,

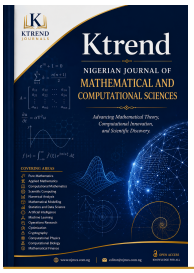
$$C_{\Gamma}(\alpha) = \Omega.$$

It follows from Definition 2.1 that Γ is connected.

Combining (2.1) and (2.1), we obtain

$$\Gamma \text{ is connected} \iff \langle H, g \rangle = W.$$

This completes the proof. □



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2.2. Degree of a Generalized Distance Graph

Definition 2.3 (Generalized Distance Graph)**.** Let

$$S \subseteq \{1, \ldots, n\}.$$

Define the graph $\Gamma_S = (\Omega, E_S)$ by

$$E_S = \{\{\alpha, \beta\} : \alpha, \beta \in \Omega,\ \alpha \neq \beta,\ d_H(\alpha, \beta) \in S\}. \tag{2.1}$$

Theorem 2.4 (Degree of a Generalized Distance Graph)**.** *Let Γ_S be the generalized distance graph defined in Definition 2.3. Then Γ_S is regular of degree*

$$\boxed{\deg(\Gamma_S) = \sum_{k \in S} \binom{n}{k} (m-1)^k}. \tag{2.2}$$

Moreover,

$$|V(\Gamma_S)| = m^n. \tag{2.3}$$

Proof. Let

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega$$

be arbitrary. By (2.1), the neighbourhood of α is

$$N_{\Gamma_S}(\alpha) = \{\beta \in \Omega : d_H(\alpha, \beta) \in S\}. \tag{2.4}$$

Since $S \subseteq \{1, \ldots, n\}$, every β in (2.4) differs from α in at least one coordinate.

For each $k \in S$, define

$$N_k(\alpha) = \{\beta \in \Omega : d_H(\alpha, \beta) = k\}. \tag{2.5}$$

Then

$$N_{\Gamma_S}(\alpha) = \bigcup_{k \in S} N_k(\alpha). \tag{2.6}$$

If $k, \ell \in S$ and $k \neq \ell$, then

$$N_k(\alpha) \cap N_\ell(\alpha) = \varnothing,$$

because a vertex cannot have two different Hamming distances from the same fixed vertex. Hence the union in (2.6) is disjoint, and

$$\deg_{\Gamma_S}(\alpha) = \sum_{k \in S} |N_k(\alpha)|. \tag{2.7}$$

We now calculate $|N_k(\alpha)|$.

Step 1: Choose the coordinates that differ.

For $\beta \in N_k(\alpha)$, exactly k coordinates satisfy $\beta_i \neq \alpha_i$. The set of these coordinates can be chosen in

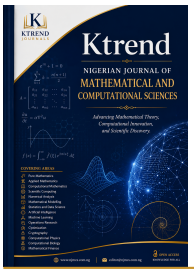
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.8}$$

ways.

Step 2: Choose the entries in the differing coordinates.

For each selected coordinate i, the entry β_i must belong to $X \setminus \{\alpha_i\}$. Since $|X| = m$,

$$|X \setminus \{\alpha_i\}| = m - 1. \tag{2.9}$$



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The choices in the k selected coordinates are independent. Therefore, the number of possible entries in those coordinates is

$$\underbrace{(m-1)(m-1)\cdots(m-1)}_{k \text{ factors}} = (m-1)^k. \tag{2.10}$$

In each of the remaining $n-k$ coordinates, we must have

$$\beta_i = \alpha_i,$$

so there is exactly one choice per coordinate.

Step 3: Count vertices at distance k.

Combining (2.8) and (2.10), the multiplication principle gives

$$|N_k(\alpha)| = \binom{n}{k}(m-1)^k. \tag{2.11}$$

Step 4: Calculate the degree.

Substituting (2.11) into (2.7), we obtain

$$\deg_{\Gamma_S}(\alpha) = \sum_{k\in S} |N_k(\alpha)| \tag{2.12}$$

$$= \sum_{k\in S} \binom{n}{k}(m-1)^k. \tag{2.13}$$

The right-hand side of (2.13) depends only on n, m, and S, and is independent of the choice of $\alpha \in \Omega$. Thus

$$\deg_{\Gamma_S}(\alpha) = \deg_{\Gamma_S}(\beta) \qquad \forall\, \alpha, \beta \in \Omega.$$

Consequently, Γ_S is regular, and its degree is given by (2.2).

Finally, since $|X| = m$ and $\Omega = X^n$, the multiplication principle yields

$$|V(\Gamma_S)| = |\Omega| = |X|^n = m^n, \tag{2.14}$$

which proves (2.3). Therefore the theorem holds. □

2.3. Hamming Distance Graphs

Definition 2.5 (Hamming Distance-k Graph)**.** Let

$$X = \{1, 2, \ldots, m\}, \qquad m \geq 2,$$

and let $\Omega = X^n$, where $n \geq 1$. For $\alpha, \beta \in \Omega$, define

$$d_H(\alpha, \beta) = |\{i \in \{1, \ldots, n\} : \alpha_i \neq \beta_i\}|.$$

For each $k \in \{1, \ldots, n\}$, define

$$\Gamma_k = (\Omega, E_k),$$

where

$$\{\alpha, \beta\} \in E_k \iff d_H(\alpha, \beta) = k. \tag{2.15}$$

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Theorem 2.6 (Distance Graphs). *For each $k \in \{1, \ldots, n\}$, the graph Γ_k defined in Definition 2.5 is regular, with*

$$\boxed{\deg(\Gamma_k) = \binom{n}{k}(m-1)^k}. \tag{2.16}$$

Furthermore,

$$|V(\Gamma_k)| = m^n. \tag{2.17}$$

In particular,

$$\Gamma_1 = H(n, m), \tag{2.18}$$

where $H(n, m)$ is the Hamming graph.

Proof. Let

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega$$

be arbitrary. By (2.15), the neighbourhood of α in Γ_k is

$$N_{\Gamma_k}(\alpha) = \{\beta \in \Omega : d_H(\alpha, \beta) = k\}. \tag{2.19}$$

Thus

$$\deg_{\Gamma_k}(\alpha) = |N_{\Gamma_k}(\alpha)|. \tag{2.20}$$

We count the elements of $N_{\Gamma_k}(\alpha)$ as follows.

Step 1: Selection of differing coordinates.
For $\beta \in N_{\Gamma_k}(\alpha)$, exactly k coordinates satisfy $\beta_i \neq \alpha_i$. Hence the set

$$I = \{i \in \{1, \ldots, n\} : \beta_i \neq \alpha_i\}$$

satisfies $|I| = k$.

The number of possible choices for I is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2.21}$$

Step 2: Selection of alternative symbols.
Fix a set I with $|I| = k$. For every $i \in I$, the coordinate β_i must satisfy

$$\beta_i \in X \setminus \{\alpha_i\}.$$

Since $|X| = m$, it follows that

$$|X \setminus \{\alpha_i\}| = m - 1. \tag{2.22}$$

The choices in distinct coordinates are independent. Therefore, the number of possible assignments to the coordinates in I is

$$\prod_{i \in I} |X \setminus \{\alpha_i\}| = \prod_{i \in I}(m-1) = (m-1)^k. \tag{2.23}$$

For every $j \notin I$, the coordinate must remain unchanged:

$$\beta_j = \alpha_j.$$

Thus each coordinate outside I contributes exactly one choice.

Step 3: Counting the neighbourhood.

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By the multiplication principle, equations (2.21) and (2.23) yield

$$|N_{\Gamma_k}(\alpha)| = \binom{n}{k}(m-1)^k \tag{2.24}$$
$$= \frac{n!}{k!(n-k)!}(m-1)^k.$$

Consequently, by (2.20),

$$\deg_{\Gamma_k}(\alpha) = \binom{n}{k}(m-1)^k. \tag{2.25}$$

Since the right-hand side of (2.25) is independent of α, we have

$$\deg_{\Gamma_k}(\alpha) = \deg_{\Gamma_k}(\beta) \qquad \forall\, \alpha, \beta \in \Omega.$$

Therefore, Γ_k is regular, and (2.16) follows.

Step 4: Number of vertices.
Since $\Omega = X^n$ and $|X| = m$, the multiplication principle gives

$$|V(\Gamma_k)| = |\Omega| = |X|^n = m^n. \tag{2.26}$$

This proves (2.17).

Step 5: Identification of Γ_1.
Setting $k = 1$ in (2.15), we obtain

$$\{\alpha, \beta\} \in E_1 \iff d_H(\alpha, \beta) = 1.$$

Thus two distinct vertices are adjacent precisely when they differ in exactly one coordinate. This is the defining adjacency condition of the Hamming graph $H(n, m)$. Hence

$$\Gamma_1 = H(n, m),$$

as asserted in (2.18). The proof is complete. □

2.4. Complement of a Distance Graph

Theorem 2.7 (Complement of a Distance Graph)**.** *Let $X = \{1, 2, \ldots, m\}$, where $m \geq 2$, and let*

$$\Omega = X^n, \qquad n \geq 1.$$

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in Ω, define the Hamming distance by

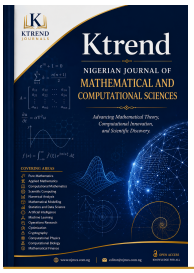
$$d_H(\alpha, \beta) = |\{i \in \{1, \ldots, n\} : \alpha_i \neq \beta_i\}|. \tag{2.27}$$

Let $\Gamma_k = (\Omega, E_k)$, where $1 \leq k \leq n$, be the graph in which

$$\{\alpha, \beta\} \in E_k \iff d_H(\alpha, \beta) = k. \tag{2.28}$$

Then the complement $\overline{\Gamma_k}$ has vertex set Ω, and

$$\{\alpha, \beta\} \in E(\overline{\Gamma_k}) \iff \alpha \neq \beta \quad \text{and} \quad d_H(\alpha, \beta) \neq k. \tag{2.29}$$



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Moreover, $\overline{\Gamma_k}$ *is regular of degree*

$$\deg(\overline{\Gamma_k}) = m^n - 1 - \binom{n}{k}(m-1)^k. \tag{2.30}$$

More generally, if $S \subseteq \{1, \ldots, n\}$ *and* Γ_S *is defined by*

$$\{\alpha, \beta\} \in E(\Gamma_S) \iff d_H(\alpha, \beta) \in S, \tag{2.31}$$

then

$$\overline{\Gamma_S} = \Gamma_{\{1,\ldots,n\}\setminus S}. \tag{2.32}$$

Proof. Fix an arbitrary vertex

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega.$$

Since $|X| = m$, the number of vertices in $\Omega = X^n$ is

$$|\Omega| = |X|^n = m^n. \tag{2.33}$$

Consequently, the number of vertices distinct from α is

$$|\Omega \setminus \{\alpha\}| = m^n - 1. \tag{2.34}$$

We first determine the degree of α in Γ_k. A vertex β is adjacent to α precisely when

$$d_H(\alpha, \beta) = k.$$

By the definition of Hamming distance in (2.27), this means that α and β differ in exactly k coordinates.

There are

$$\binom{n}{k}$$

ways to select the k coordinates in which α and β differ. At each selected coordinate i, the value β_i can be chosen from $X \setminus \{\alpha_i\}$, which contains $m-1$ elements. Thus, for every selection of k coordinates, there are

$$(m-1)^k$$

possible choices for the entries of β in those coordinates. The remaining $n-k$ coordinates must agree with α.

It follows that

$$\deg_{\Gamma_k}(\alpha) = \binom{n}{k}(m-1)^k. \tag{2.35}$$

Since the right-hand side of (2.35) is independent of α, the graph Γ_k is regular.

By the definition of the complement, two distinct vertices are adjacent in $\overline{\Gamma_k}$ if and only if they are not adjacent in Γ_k. Hence

$$\{\alpha, \beta\} \in E(\overline{\Gamma_k}) \iff \alpha \neq \beta \text{ and } d_H(\alpha, \beta) \neq k,$$

which proves (2.29).

Among the $m^n - 1$ vertices distinct from α, exactly $\binom{n}{k}(m-1)^k$ are adjacent to α in Γ_k. All the remaining vertices are adjacent to α in $\overline{\Gamma_k}$. Therefore,

$$\deg_{\overline{\Gamma_k}}(\alpha) = (m^n - 1) - \deg_{\Gamma_k}(\alpha) \tag{2.36}$$

$$= m^n - 1 - \binom{n}{k}(m-1)^k. \tag{2.37}$$

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The expression in (2.37) does not depend on α. Thus $\overline{\Gamma_k}$ is regular, and (2.30) follows.

Finally, let $S \subseteq \{1, \ldots, n\}$. For distinct $\alpha, \beta \in \Omega$,

$$\{\alpha, \beta\} \in E(\overline{\Gamma_S}) \iff d_H(\alpha, \beta) \notin S.$$

Since $\alpha \neq \beta$, we have

$$1 \leq d_H(\alpha, \beta) \leq n.$$

Therefore,

$$d_H(\alpha, \beta) \notin S \iff d_H(\alpha, \beta) \in \{1, \ldots, n\} \setminus S.$$

By the definition of the generalized distance graph in (2.31), this is equivalent to

$$\{\alpha, \beta\} \in E(\Gamma_{\{1,\ldots,n\}\setminus S}).$$

Both graphs have vertex set Ω, so their edge sets coincide. Hence

$$\overline{\Gamma_S} = \Gamma_{\{1,\ldots,n\}\setminus S},$$

as required. □

2.5. Connectivity and Diameter of the Hamming Graph

Definition 2.8 (Graph Distance and Diameter)**.** Let Γ be a connected graph. The distance $d_\Gamma(u, v)$ between vertices u and v is the minimum length of a path joining them. The diameter of Γ is

$$\operatorname{diam}(\Gamma) = \max_{u,v \in V(\Gamma)} d_\Gamma(u, v).$$

Theorem 2.9 (Connectivity and Diameter of the Hamming Graph)**.** *Let $m \geq 2$ and $n \geq 1$ be integers. Then the Hamming graph $H(n, m)$ is connected. Moreover, for all $\alpha, \beta \in X^n$,*

$$\boxed{d_{H(n,m)}(\alpha, \beta) = d_H(\alpha, \beta) = |\{i : \alpha_i \neq \beta_i\}|.}$$

Consequently,

$$\boxed{\operatorname{diam}(H(n, m)) = n.}$$

In addition, $H(n, m)$ has m^n vertices and is a regular graph of degree $n(m - 1)$.

Proof. We divide the proof into four parts.

Part I: The number of vertices.

Since

$$\Omega = X^n,$$

and $|X| = m$, the multiplication principle gives

$$|V(H(n, m))| = |\Omega| = |X|^n = m^n.$$

Thus, $H(n, m)$ has exactly m^n vertices.

Part II: The exact distance formula.

Let

$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad \beta = (\beta_1, \ldots, \beta_n)$$

be arbitrary vertices of $H(n, m)$. Define

$$I(\alpha, \beta) = \{i \in \{1, \ldots, n\} : \alpha_i \neq \beta_i\}$$

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and put

$$k = |I(\alpha, \beta)| = d_H(\alpha, \beta).$$

We prove that

$$d_{H(n,m)}(\alpha, \beta) = k.$$

First, consider any path

$$\alpha = v^{(0)}, v^{(1)}, \ldots, v^{(\ell)} = \beta$$

from α to β. By the definition of the Hamming graph, each consecutive pair $v^{(j-1)}, v^{(j)}$ differs in exactly one coordinate.

For every coordinate $i \in I(\alpha, \beta)$, the initial and final entries are different:

$$\alpha_i \neq \beta_i.$$

Therefore, along the path, the i-th coordinate must change at least once. Since a single edge changes only one coordinate, at least k edges are required. Hence every path from α to β has length at least k, and consequently

$$d_{H(n,m)}(\alpha, \beta) \geq k. \tag{1}$$

Conversely, list the coordinates in $I(\alpha, \beta)$ as

$$I(\alpha, \beta) = \{i_1, i_2, \ldots, i_k\}.$$

Starting from α, change coordinate i_1 to β_{i_1}, then coordinate i_2 to β_{i_2}, and continue in this manner until coordinate i_k has been changed to its corresponding entry in β.

This produces the sequence

$$\alpha = v^{(0)}, v^{(1)}, \ldots, v^{(k)} = \beta,$$

where each consecutive pair differs in exactly one coordinate. Thus every consecutive pair is adjacent in $H(n, m)$, so the sequence is a path of length k. It follows that

$$d_{H(n,m)}(\alpha, \beta) \leq k. \tag{2}$$

Combining (1) and (2), we obtain

$$\boxed{d_{H(n,m)}(\alpha, \beta) = d_H(\alpha, \beta) = k.} \tag{3}$$

Part III: Connectivity.

Since α and β were arbitrary vertices, equation (3) shows that there is a path joining every pair of vertices. Therefore, $H(n, m)$ is connected.

Part IV: The diameter.

For any $\alpha, \beta \in X^n$, there are only n coordinates. Hence

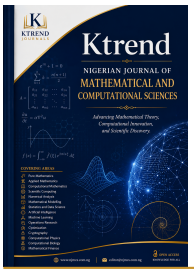
$$d_H(\alpha, \beta) \leq n.$$

By equation (3),

$$d_{H(n,m)}(\alpha, \beta) \leq n.$$

Taking the maximum over all pairs of vertices gives

$$\operatorname{diam}(H(n, m)) \leq n. \tag{4}$$



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It remains to establish the reverse inequality. Since $m \geq 2$, there exist distinct elements $a, b \in X$. Consider

$$u = (a, a, \ldots, a), \qquad v = (b, b, \ldots, b).$$

These vertices differ in all n coordinates. Therefore,

$$d_H(u, v) = n.$$

By equation (3),

$$d_{H(n,m)}(u, v) = n.$$

Consequently,

$$\operatorname{diam}(H(n, m)) \geq n. \tag{5}$$

Combining (4) and (5), we conclude that

$$\boxed{\operatorname{diam}(H(n, m)) = n.}$$

Finally, we determine the degree of the graph. Let

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in X^n.$$

A neighbour of α is obtained by choosing one of the n coordinates and replacing its entry by one of the other $m - 1$ elements of X. These choices give distinct neighbours, and every neighbour arises in this way. Hence

$$\deg(\alpha) = n(m - 1).$$

Since this value is independent of α, the graph is regular of degree $n(m - 1)$. This completes the proof. □

Remark 2.10. The vertex set X^n is a Cartesian power. The Hamming graph is the Cartesian product of n copies of the complete graph K_m:

$$\boxed{H(n, m) = \underbrace{K_m \square K_m \square \cdots \square K_m}_{n \text{ factors}}.}$$

Here, $\square$ denotes the Cartesian product of graphs.

If G and F are graphs, their Cartesian product $G \square F$ has vertex set $V(G) \times V(F)$, with

$$(u, a) \sim (v, b)$$

if and only if either

$$u = v \quad \text{and} \quad a \sim_F b,$$

or

$$a = b \quad \text{and} \quad u \sim_G v.$$

Applying this construction repeatedly gives precisely the adjacency rule for $H(n, m)$.

Furthermore, the symmetric group S_m acts on X, and the wreath product $S_m \wr S_n$ acts naturally on X^n in product action. This action preserves Hamming distance, because permutations of the symbols preserve equality and inequality of entries, while permutations of coordinates preserve the number of coordinates in which two tuples differ.

Thus, the Hamming graph is compatible with the algebraic structure of the wreath product $S_m \wr S_n$.

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Corollary 2.11 (Regularity and Number of Edges)**.** *For $m \geq 2$ and $n \geq 1$, the Hamming graph $H(n, m)$ has*

$$|V(H(n, m))| = m^n$$

vertices and degree

$$\deg(H(n, m)) = n(m - 1).$$

Its number of edges is

$$\boxed{|E(H(n, m))| = \frac{m^n n(m - 1)}{2}.}$$

Proof. The vertex and degree formulas follow from Theorem 2.9. By the handshaking lemma,

$$2|E(H(n, m))| = \sum_{v \in V(H(n,m))} \deg(v).$$

Since every vertex has degree $n(m - 1)$, we obtain

$$2|E(H(n, m))| = m^n n(m - 1),$$

and hence

$$|E(H(n, m))| = \frac{m^n n(m - 1)}{2}.$$

□

Corollary 2.12 (Distance Distribution)**.** *Fix any vertex $\alpha \in X^n$. For each integer k with $0 \leq k \leq n$, the number of vertices at distance k from α is*

$$\boxed{\binom{n}{k}(m - 1)^k.}$$

Proof. To construct a vertex at distance k from α, first choose the k coordinates in which it differs from α. There are $\binom{n}{k}$ ways to choose these coordinates. In each selected coordinate, there are $m - 1$ possible entries different from the corresponding entry of α. The other $n - k$ coordinates must remain unchanged. Therefore, the required number is

$$\binom{n}{k}(m - 1)^k.$$

□

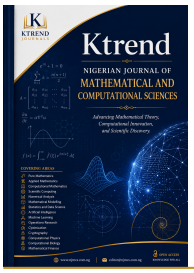
Definition 2.13 (Cycle and Girth)**.** Let $\Gamma = (V, E)$ be a simple graph. A cycle of length r is a sequence of distinct vertices

$$v_1, v_2, \ldots, v_r$$

such that

$$v_i \sim v_{i+1} \quad (1 \leq i < r), \qquad v_r \sim v_1.$$

The girth of Γ, denoted by $\text{girth}(\Gamma)$, is the length of its shortest cycle. If Γ contains no cycle, its girth is defined to be infinity.



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Theorem 2.14 (Girth of the Hamming Graph)**.** *Let $m \geq 2$ and $n \geq 1$ be integers. Then the Hamming graph $H(n, m)$ satisfies*

$$\boxed{\operatorname{girth}(H(n,m)) = \begin{cases} 3, & m \geq 3,\ n \geq 1, \\ 4, & m = 2,\ n \geq 2, \\ \infty, & m = 2,\ n = 1. \end{cases}}$$

In particular, for $m \geq 3$, the graph contains a triangle, whereas for $m = 2$ and $n \geq 2$, it is a bipartite graph whose shortest cycle has length four.

Proof. Recall that the vertex set of $H(n, m)$ is

$$V(H(n,m)) = X^n, \qquad X = \{1, 2, \ldots, m\},$$

and two distinct vertices are adjacent precisely when they differ in exactly one coordinate.

We consider three cases.

Case 1: $m \geq 3$ and $n \geq 1$.

Since $m \geq 3$, choose three distinct symbols $a, b, c \in X$. Let

$$u = (a, a, \ldots, a),$$

and define

$$v = (b, a, \ldots, a), \qquad w = (c, a, \ldots, a).$$

The vertices u, v, w are distinct. Moreover, u and v differ only in the first coordinate, so $u \sim v$. Similarly, $u \sim w$ and $v \sim w$, because $b \neq c$ and the two latter vertices differ only in the first coordinate.

Thus,

$$u \sim v, \qquad v \sim w, \qquad w \sim u.$$

Consequently, u, v, w form a cycle of length three. Hence

$$\operatorname{girth}(H(n,m)) \leq 3.$$

Since $H(n, m)$ is a simple graph, it has neither loops nor cycles of length two. Therefore, every cycle has length at least three. It follows that

$$\boxed{\operatorname{girth}(H(n,m)) = 3.}$$

Case 2: $m = 2$ and $n \geq 2$.

Let $X = \{0, 1\}$. The vertices of $H(n, 2)$ are binary strings of length n, and two vertices are adjacent if they differ in exactly one coordinate.

Define the parity of a vertex

$$x = (x_1, \ldots, x_n) \in \{0, 1\}^n$$

by

$$p(x) \equiv x_1 + x_2 + \cdots + x_n \pmod 2.$$

Partition the vertex set into

$$V_0 = \{x \in \{0, 1\}^n : p(x) = 0\},$$

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and

$$V_1 = \{x \in \{0,1\}^n : p(x) = 1\}.$$

If $x \sim y$, then x and y differ in exactly one coordinate. Changing a binary coordinate from 0 to 1, or from 1 to 0, changes the parity. Hence every edge joins a vertex of V_0 to a vertex of V_1.

Therefore,

$$V(H(n,2)) = V_0 \cup V_1, \qquad V_0 \cap V_1 = \varnothing,$$

and $H(n,2)$ is bipartite. A bipartite graph contains no odd cycle; in particular, it contains no triangle. Thus

$$\text{girth}(H(n,2)) \geq 4$$

whenever a cycle exists.

It remains to exhibit a cycle of length four. Since $n \geq 2$, consider the vertices

$$u = (0,0,0,\ldots,0),$$

$$v = (1,0,0,\ldots,0),$$

$$w = (1,1,0,\ldots,0),$$

and

$$z = (0,1,0,\ldots,0).$$

They are distinct, and consecutive vertices differ in exactly one coordinate. Therefore,

$$u \sim v, \qquad v \sim w, \qquad w \sim z, \qquad z \sim u.$$

These vertices form a four-cycle. Consequently,

$$\text{girth}(H(n,2)) \leq 4.$$

Combining the two inequalities gives

$$\boxed{\text{girth}(H(n,2)) = 4.}$$

Case 3: $m = 2$ **and** $n = 1$**.**

The graph $H(1,2)$ has exactly two vertices, 0 and 1, joined by one edge. It is the complete graph K_2, which has no cycle. By the convention in the definition of girth,

$$\boxed{\text{girth}(H(1,2)) = \infty.}$$

□

Remark 2.15. The three cases establish the theorem.

The Hamming graph has the Cartesian-product representation

$$H(n,m) = \underbrace{K_m \square \cdots \square K_m}_{n \text{ factors}}.$$

When $m \geq 3$, each factor K_m contains a triangle. Fixing all coordinates except one embeds a copy of K_m into $H(n,m)$. Therefore, the triangle structure of the factor is preserved in the Cartesian product, and the girth is three.

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When $m = 2$, each factor is K_2, and hence

$$H(n, 2) = \underbrace{K_2 \square \cdots \square K_2}_{n \text{ factors}}.$$

This is the n-dimensional hypercube graph. Its vertices are binary n-tuples, and its bipartition is determined by the parity of the sum of the coordinates. For $n \geq 2$, changing two different coordinates in either order gives a four-cycle.

Definition 2.16 (Bipartite Graph)**.** A graph $\Gamma = (V, E)$ is called bipartite if its vertex set can be partitioned into two disjoint sets V_0 and V_1 such that every edge has one endpoint in V_0 and the other in V_1. In other words,

$$V = V_0 \cup V_1, \qquad V_0 \cap V_1 = \varnothing,$$

and no two vertices in the same part are adjacent.

Theorem 2.17 (Bipartiteness of the Hamming Graph)**.** *Let $n \geq 1$ and $m \geq 2$. The Hamming graph $H(n, m)$ is bipartite if and only if*

$$m = 2.$$

Proof. We prove both directions.

Case 1: $m = 2$**.**

Suppose that $m = 2$. Then

$$X = \{1, 2\} \quad \text{and} \quad V(H(n, 2)) = \{1, 2\}^n.$$

For each vertex

$$u = (u_1, u_2, \ldots, u_n),$$

define

$$p(u) = |\{i \in \{1, \ldots, n\} : u_i = 2\}|.$$

Thus, $p(u)$ is the number of coordinates of u whose entries are equal to 2.

Partition the vertex set into

$$V_0 = \{u \in \{1, 2\}^n : p(u) \text{ is even}\}$$

and

$$V_1 = \{u \in \{1, 2\}^n : p(u) \text{ is odd}\}.$$

Every vertex belongs to exactly one of these sets. Consequently,

$$V_0 \cap V_1 = \varnothing, \qquad V_0 \cup V_1 = \{1, 2\}^n.$$

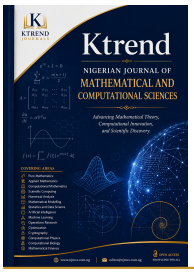
Now let u and v be adjacent vertices of $H(n, 2)$. By the definition of the Hamming graph, they differ in exactly one coordinate. Since the only possible symbols are 1 and 2, the differing coordinate changes from 1 to 2 or from 2 to 1.

Therefore, the number of coordinates equal to 2 changes by exactly one. Hence

$$p(v) = p(u) + 1 \quad \text{or} \quad p(v) = p(u) - 1.$$

It follows that $p(u)$ and $p(v)$ have opposite parity. Thus, one of u, v belongs to V_0 and the other belongs to V_1.

Therefore, every edge joins a vertex in V_0 to a vertex in V_1. Hence $H(n, 2)$ is bipartite.



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Case 2: $m \geq 3$**.**

Suppose that $m \geq 3$. Choose three distinct elements $a, b, c \in X$. Fix arbitrary entries $x_2, \ldots, x_n \in X$ for all coordinates other than the first, and consider the three vertices

$$u = (a, x_2, \ldots, x_n),$$

$$v = (b, x_2, \ldots, x_n),$$

and

$$w = (c, x_2, \ldots, x_n).$$

The vertices u and v differ only in their first coordinate, so $u \sim v$. Similarly,

$$v \sim w \quad \text{and} \quad w \sim u.$$

Consequently, these three vertices form a triangle:

$$u \sim v \sim w \sim u.$$

However, a bipartite graph cannot contain a triangle. Indeed, if u belongs to one part, say V_0, then its neighbour v must belong to V_1. Since v is adjacent to w, the vertex w must belong to V_0. But w is also adjacent to u, which would place an edge between two vertices in V_0, a contradiction.

Therefore, $H(n, m)$ is not bipartite whenever $m \geq 3$.

Combining the two cases, we conclude that

$$\boxed{H(n, m) \text{ is bipartite if and only if } m = 2.}$$

□

2.6. Cartesian Products and Coordinate Graphs

Definition 2.18 (Cartesian Product of Graphs)**.** Let $F = (X, E)$ be a simple graph. The Cartesian power $F^{\square n}$ is the graph whose vertex set is

$$V(F^{\square n}) = X^n.$$

Two distinct vertices

$$u = (u_1, \ldots, u_n), \qquad v = (v_1, \ldots, v_n)$$

are adjacent if and only if they differ in exactly one coordinate, say coordinate i, and the entries u_i and v_i are adjacent in F.

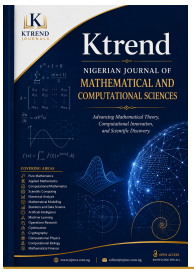
Definition 2.19 (Regular Graph)**.** A graph is called r-regular if every vertex has exactly r neighbours.

Definition 2.20 (Graph Diameter)**.** For a connected graph F, the distance $d_F(x, y)$ between vertices x and y is the length of a shortest path joining them. The diameter is

$$\operatorname{diam}(F) = \max_{x,y \in V(F)} d_F(x, y).$$

Definition 2.21 (Girth)**.** The girth of a graph is the length of its shortest cycle. If a graph has no cycles, its girth is defined to be infinity:

$$\operatorname{girth}(F) = \infty.$$



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Theorem 2.22 (Cartesian Power)**.** *Let F be a connected r-regular graph on m vertices. Then $F^{\square n}$ is connected and regular of degree*

$$\boxed{d(F^{\square n}) = nr.}$$

Moreover,

$$\boxed{\operatorname{diam}(F^{\square n}) = n \operatorname{diam}(F).}$$

If F has at least one edge and $n \geq 2$, then

$$\boxed{\operatorname{girth}(F^{\square n}) = \min\{4, \operatorname{girth}(F)\}.}$$

Here an acyclic factor is assigned infinite girth.

Proof. We prove the claims separately.

Part 1: Regularity.

Let

$$u = (u_1, u_2, \ldots, u_n) \in V(F^{\square n}).$$

Since F is r-regular, the coordinate u_i has exactly r neighbours in F.

For each of the n coordinates, we may replace u_i by any one of its r neighbours while keeping all other coordinates fixed. Each such replacement produces a neighbour of u in $F^{\square n}$.

Changes in different coordinates produce distinct neighbours. Hence the number of neighbours of u is

$$\deg_{F^{\square n}}(u) = \underbrace{r + r + \cdots + r}_{n \text{ terms}} = nr.$$

This number is independent of the choice of u. Therefore, $F^{\square n}$ is regular of degree

$$\boxed{d(F^{\square n}) = nr.}$$

Part 2: Connectivity.

Let

$$u = (u_1, \ldots, u_n), \qquad v = (v_1, \ldots, v_n)$$

be arbitrary vertices of $F^{\square n}$.

Because F is connected, for each coordinate i there exists a path in F from u_i to v_i. Write a shortest such path as

$$u_i = x_{i,0}, x_{i,1}, \ldots, x_{i,\ell_i} = v_i,$$

where

$$\ell_i = d_F(u_i, v_i).$$

Starting from u, follow the path in the first coordinate while fixing all other coordinates. Next, follow the path in the second coordinate, and continue in this way until all coordinates agree with v.

Every step changes exactly one coordinate along an edge of F, so every step is an edge of $F^{\square n}$. The resulting sequence is a path from u to v.

Since u and v were arbitrary, every pair of vertices is joined by a path. Therefore,

$$\boxed{F^{\square n} \text{ is connected.}}$$

Part 3: Diameter.

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We first establish the distance formula

$$\boxed{d_{F^{\square n}}(u,v) = \sum_{i=1}^{n} d_F(u_i, v_i).}$$

For the upper bound, the coordinate-by-coordinate construction in Part 2 gives a path of length

$$\sum_{i=1}^{n} d_F(u_i, v_i).$$

Consequently,

$$d_{F^{\square n}}(u,v) \leq \sum_{i=1}^{n} d_F(u_i, v_i).$$

For the lower bound, consider any path from u to v in $F^{\square n}$. Each edge changes exactly one coordinate. For each coordinate i, the changes in that coordinate must provide a walk in F from u_i to v_i. At least $d_F(u_i, v_i)$ changes are therefore required in coordinate i.

Summing over all coordinates, every path from u to v has length at least

$$\sum_{i=1}^{n} d_F(u_i, v_i).$$

Thus,

$$d_{F^{\square n}}(u,v) \geq \sum_{i=1}^{n} d_F(u_i, v_i).$$

The two inequalities establish the distance formula.

Let

$$D = \operatorname{diam}(F).$$

For every pair $u, v \in X^n$, the distance formula gives

$$d_{F^{\square n}}(u,v) = \sum_{i=1}^{n} d_F(u_i, v_i) \leq \sum_{i=1}^{n} D = nD.$$

Therefore,

$$\operatorname{diam}(F^{\square n}) \leq nD.$$

Since F is finite and connected, there exist vertices $a, b \in X$ satisfying

$$d_F(a,b) = D.$$

Consider the vertices

$$\alpha = (a, a, \ldots, a), \qquad \beta = (b, b, \ldots, b).$$

Their distance is

$$d_{F^{\square n}}(\alpha, \beta) = \sum_{i=1}^{n} d_F(a,b) = nD.$$

Hence the diameter is at least nD. Combining both bounds yields

$$\boxed{\operatorname{diam}(F^{\square n}) = n \operatorname{diam}(F).}$$

Part 4: Girth.

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Assume that F has at least one edge and $n \geq 2$. Put

$$g = \text{girth}(F).$$

First, every cycle contained in a single coordinate copy of F has length at least g. Therefore, if F contains a cycle of length g, that cycle also appears in $F^{\square n}$ by fixing all other coordinates. Thus, when g is finite,

$$\text{girth}(F^{\square n}) \leq g.$$

Next, we show that $F^{\square n}$ contains a four-cycle. Choose an edge $a \sim b$ in F and fix any vertex $c \in V(F)$. Since $n \geq 2$, consider

$$u = (a, c, \ldots), \qquad v = (b, c, \ldots),$$

$$w = (b, b, c, \ldots), \qquad z = (a, b, c, \ldots),$$

where the displayed tuples use the first two coordinates, and all remaining coordinates are fixed at c.

These vertices form a four-cycle: the first and third edges change the first coordinate, while the second and fourth edges change the second coordinate. Consequently,

$$\text{girth}(F^{\square n}) \leq 4.$$

It remains to show that no cycle shorter than $\min\{4, g\}$ can occur. A cycle contained in one coordinate copy of F has length at least g. A cycle that changes more than one coordinate cannot have length three: each coordinate must return to its starting value, and a closed walk that changes coordinates requires at least two changes in each coordinate it uses. Hence a cycle involving at least two coordinates has length at least four.

It follows that every cycle has length at least $\min\{4, g\}$. Together with the upper bounds, this gives

$$\boxed{\text{girth}(F^{\square n}) = \min\{4, \text{girth}(F)\}.}$$

If F is acyclic, then $g = \infty$, and the same argument gives girth 4 for the Cartesian power when $n \geq 2$.

This completes the proof. □

2.7. Cayley Graph Properties

Definition 2.23 (Inverse-closed subset)**.** Let P be a group with identity element e. A subset $T \subseteq P$ is called inverse-closed if

$$t \in T \implies t^{-1} \in T.$$

Throughout this section, we assume that

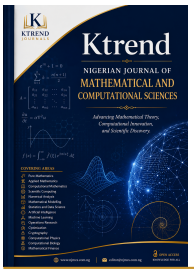
$$e \notin T \qquad \text{and} \qquad T = T^{-1}.$$

Definition 2.24 (Cayley graph)**.** Let P be a group and let $T \subseteq P$ be an inverse-closed subset not containing the identity. The Cayley graph $\text{Cay}(P, T)$ is the graph whose vertex set is P, with two distinct vertices $u, v \in P$ adjacent precisely when

$$u^{-1}v \in T.$$

Equivalently,

$$u \sim v \iff u^{-1}v \in T.$$



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Definition 2.25 (Word length)**.** Let $P = \langle T \rangle$. For $g \in P$, the word length of g with respect to T, denoted by $\ell_T(g)$, is the smallest non-negative integer k for which

$$g = t_1 t_2 \cdots t_k, \qquad t_i \in T.$$

The identity has word length $\ell_T(e) = 0$.

Definition 2.26 (Diameter)**.** The diameter of a connected graph Γ is

$$\operatorname{diam}(\Gamma) = \max_{u,v \in V(\Gamma)} d(u, v),$$

where $d(u, v)$ denotes the length of a shortest path between u and v.

Theorem 2.27 (Cayley Graph Properties)**.** *Let P be a finite group and let $T \subseteq P \setminus \{e\}$ satisfy $T = T^{-1}$. Then $\mathrm{Cay}(P, T)$ is a simple, undirected, regular graph of degree $|T|$. Furthermore, it is connected if and only if*

$$\langle T \rangle = P.$$

If the graph is connected, then

$$\operatorname{diam}(\mathrm{Cay}(P, T)) = \max_{g \in P} \ell_T(g).$$

Proof. We prove the assertions separately.

Step 1: The graph has no loops.
Suppose that a vertex $u \in P$ were adjacent to itself. By the definition of adjacency, this would require

$$u^{-1}u = e \in T.$$

However, $e \notin T$. This is a contradiction. Therefore, the graph has no loops.

Step 2: The graph is undirected.
Suppose that $u \sim v$. Then

$$u^{-1}v \in T.$$

Since $T = T^{-1}$, it follows that

$$(u^{-1}v)^{-1} = v^{-1}u \in T.$$

Consequently, $v \sim u$. Thus adjacency is symmetric, and the graph is undirected.
Together, Steps 1 and 2 show that the graph is simple and undirected.

Step 3: The graph is regular of degree $|T|$.
Fix an arbitrary vertex $u \in P$. By the definition of the Cayley graph, the neighbours of u are precisely the elements of the set

$$N(u) = \{ut : t \in T\}.$$

Indeed, for every $t \in T$,

$$u^{-1}(ut) = t \in T,$$

so $u \sim ut$.
Conversely, if v is a neighbour of u, then $u^{-1}v \in T$. Writing $t = u^{-1}v$, we obtain $v = ut$ for some $t \in T$.
The map

$$\varphi : T \longrightarrow N(u), \qquad \varphi(t) = ut,$$

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is bijective. It is surjective by the description of $N(u)$, and it is injective because

$$ut_1 = ut_2 \implies t_1 = t_2.$$

Therefore,

$$|N(u)| = |T|.$$

Since u was arbitrary, every vertex has degree $|T|$. Hence the graph is regular of degree $|T|$.

Step 4: The connectivity criterion.

First, suppose that $\mathrm{Cay}(P, T)$ is connected. Let $g \in P$ be arbitrary. There is a path from the identity e to g, say

$$e = v_0, v_1, \ldots, v_k = g.$$

For every $i = 1, \ldots, k$, adjacency gives

$$v_{i-1}^{-1} v_i \in T.$$

Put $t_i = v_{i-1}^{-1} v_i$. Then

$$v_i = v_{i-1} t_i.$$

Applying this identity successively yields

$$g = v_k = t_1 t_2 \cdots t_k.$$

Thus $g \in \langle T \rangle$. Since g was arbitrary, $P \subseteq \langle T \rangle$. The reverse inclusion is automatic, so

$$\langle T \rangle = P.$$

Conversely, suppose that $\langle T \rangle = P$. Let $g \in P$. Since T generates P, there exist $t_1, \ldots, t_k \in T$ such that

$$g = t_1 t_2 \cdots t_k.$$

Consider the sequence

$$e,\ t_1,\ t_1 t_2,\ \ldots,\ t_1 t_2 \cdots t_k = g.$$

Each consecutive pair is adjacent because

$$(t_1 \cdots t_{i-1})^{-1} (t_1 \cdots t_i) = t_i \in T.$$

Hence there is a path from e to g. Every vertex is therefore reachable from e, and the graph is connected.

We conclude that

$$\mathrm{Cay}(P, T) \text{ is connected} \quad \iff \quad \langle T \rangle = P.$$

Step 5: The diameter formula.

Assume that the graph is connected. By the preceding step, T generates P, so every element has finite word length with respect to T.

We first show that

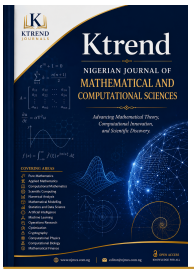
$$d(e, g) = \ell_T(g) \qquad \text{for every } g \in P.$$

Suppose that

$$g = t_1 t_2 \cdots t_k, \qquad t_i \in T.$$

The construction in Step 4 gives a path of length k from e to g. Taking a representation of minimum length gives

$$d(e, g) \leq \ell_T(g).$$



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Conversely, let

$$e = v_0, v_1, \ldots, v_k = g$$

be a shortest path from e to g. As shown in Step 4, there exist $t_1, \ldots, t_k \in T$ such that

$$g = t_1 t_2 \cdots t_k.$$

Consequently, $\ell_T(g) \leq k = d(e, g)$. Combining the two inequalities gives

$$d(e, g) = \ell_T(g).$$

Now let $u, v \in P$. Left multiplication by u^{-1} preserves adjacency: if $x \sim y$, then

$$(u^{-1}x)^{-1}(u^{-1}y) = x^{-1}y \in T.$$

Thus left multiplication is a graph automorphism. It follows that

$$d(u, v) = d(e, u^{-1}v) = \ell_T(u^{-1}v).$$

Therefore,

$$\begin{aligned} \operatorname{diam}(\operatorname{Cay}(P, T)) &= \max_{u,v \in P} d(u, v) \\ &= \max_{u,v \in P} \ell_T(u^{-1}v) \\ &= \max_{g \in P} \ell_T(g). \end{aligned}$$

The final equality holds because $u^{-1}v$ ranges over all elements of P. This proves the diameter formula and completes the proof. □

Definition 2.28 (Bipartite double-cover graph)**.** Let W act transitively on a set Ω, and let $\Delta \subseteq \Omega \times \Omega$ be an orbital. Take two disjoint copies of Ω:

$$\Omega^+ = \{u^+ : u \in \Omega\}, \qquad \Omega^- = \{v^- : v \in \Omega\}.$$

The bipartite orbital graph $B(\Delta)$ has vertex set

$$V(B(\Delta)) = \Omega^+ \cup \Omega^-,$$

with an edge between u^+ and v^- precisely when

$$(u, v) \in \Delta.$$

Theorem 2.29 (Bipartite Orbital Graph)**.** *Let W be a transitive permutation group on Ω, and let Δ be an orbital of W on $\Omega \times \Omega$. Construct*

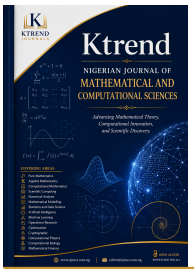
$$B(\Delta) = (\Omega^+ \cup \Omega^-, E),$$

where

$$\{u^+, v^-\} \in E \quad \Longleftrightarrow \quad (u, v) \in \Delta.$$

Then the following statements hold.
1. The graph $B(\Delta)$ is bipartite, with bipartition

$$(\Omega^+, \Omega^-).$$



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2. *If every vertex in the original orbital digraph has out-degree d^+ and in-degree d^-, then every vertex in Ω^+ has degree d^+, and every vertex in Ω^- has degree d^-.*
3. *The group W acts on $B(\Delta)$ by*

$$(u^{\pm})^w = (u^w)^{\pm}, \qquad w \in W.$$

This action preserves the two parts and is edge-transitive.

Proof. We prove the three assertions in turn.

(1) Bipartiteness.
By construction, the vertex set is the disjoint union

$$V(B(\Delta)) = \Omega^+ \cup \Omega^-, \qquad \Omega^+ \cap \Omega^- = \varnothing.$$

An edge of $B(\Delta)$ has the form

$$\{u^+, v^-\}$$

for some $u, v \in \Omega$ satisfying $(u, v) \in \Delta$. Hence every edge has one endpoint in Ω^+ and the other in Ω^-.

No edge joins two vertices in Ω^+, and no edge joins two vertices in Ω^-. Therefore,

$$\boxed{B(\Delta) \text{ is bipartite with parts } \Omega^+ \text{ and } \Omega^-.}$$

(2) Degrees in the double-cover graph.
Let $\Gamma = (\Omega, \Delta)$ be the original orbital digraph.
For a vertex $u \in \Omega$, its out-neighbourhood is

$$\Gamma^+(u) = \{v \in \Omega : (u, v) \in \Delta\}.$$

The neighbours of u^+ in $B(\Delta)$ are exactly the vertices v^- for which $(u, v) \in \Delta$. Thus

$$N_{B(\Delta)}(u^+) = \{v^- : v \in \Gamma^+(u)\}.$$

Since the correspondence $v \mapsto v^-$ is bijective,

$$\deg_{B(\Delta)}(u^+) = |\Gamma^+(u)|.$$

If the original digraph has constant out-degree d^+, then

$$\boxed{\deg_{B(\Delta)}(u^+) = d^+.}$$

Similarly, the in-neighbourhood of $v \in \Omega$ is

$$\Gamma^-(v) = \{u \in \Omega : (u, v) \in \Delta\}.$$

The neighbours of v^- in the double-cover graph are exactly the vertices u^+ for which $(u, v) \in \Delta$. Hence

$$N_{B(\Delta)}(v^-) = \{u^+ : u \in \Gamma^-(v)\},$$

and therefore

$$\deg_{B(\Delta)}(v^-) = |\Gamma^-(v)|.$$

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If the original digraph has constant in-degree d^-, then

$$\boxed{\deg_{B(\Delta)}(v^-) = d^-.}$$

This establishes the degree assertions.

(3) The action of W.

Define the action of W on the vertices of $B(\Delta)$ by

$$(u^{\pm})^w = (u^w)^{\pm}.$$

This is a group action because the original action of W on Ω satisfies the group-action identities, and the superscripts $+$ and $-$ are retained.

In particular, every element of W maps Ω^+ to itself and Ω^- to itself. Thus the action preserves both parts of the bipartition.

It remains to prove edge-transitivity. Let

$$\{u_1^+, v_1^-\}, \qquad \{u_2^+, v_2^-\}$$

be arbitrary edges of $B(\Delta)$. By the definition of the graph,

$$(u_1, v_1) \in \Delta, \qquad (u_2, v_2) \in \Delta.$$

Because Δ is an orbital, it is a single orbit of W on ordered pairs. Consequently, there exists $w \in W$ such that

$$(u_1, v_1)^w = (u_2, v_2).$$

Equivalently,

$$u_1^w = u_2, \qquad v_1^w = v_2.$$

Under the defined action, this gives

$$(u_1^+)^w = u_2^+, \qquad (v_1^-)^w = v_2^-.$$

Therefore,

$$\{u_1^+, v_1^-\}^w = \{u_2^+, v_2^-\}.$$

Since the two edges were arbitrary, W is transitive on the edges of $B(\Delta)$.

Hence the action preserves the two parts and is edge-transitive. This completes the proof. □

Definition 2.30 (Directed orbital graph)**.** Let W act transitively on Ω, and let Δ be a non-diagonal orbital. The directed orbital graph associated with Δ is

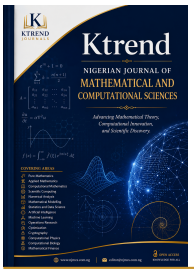
$$\Gamma = (\Omega, \Delta),$$

where the vertices are the elements of Ω, and there is an arc from u to v precisely when

$$(u, v) \in \Delta.$$

The reverse orbital is

$$\Delta^* = \{(v, u) : (u, v) \in \Delta\}.$$



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Theorem 2.31 (Directed Orbital Graph)**.** *Let W be a transitive permutation group on $\Omega = X^n$, and let*

$$\Delta = (\alpha, \beta)^W$$

be a non-diagonal orbital. Define

$$\Gamma = (\Omega, \Delta).$$

Then the following statements hold.

1. *The digraph Γ is vertex-transitive and arc-transitive under W.*
2. *Its out-degree is*

$$d^+(\Gamma) = |\beta^{W_\alpha}|,$$

where

$$W_\alpha = \{g \in W : \alpha^g = \alpha\}$$

is the stabilizer of α.

3. *Its in-degree is the size of the corresponding suborbit in the paired orbital Δ^*. More precisely, if $(\alpha, \gamma) \in \Delta^*$, then*

$$d^-(\Gamma) = |\gamma^{W_\alpha}|.$$

4. *If Δ is non-self-paired, then*

$$\Delta^* \neq \Delta.$$

Proof. We prove each assertion separately.

(1) Vertex-transitivity and arc-transitivity.
Since W acts transitively on Ω, for any two vertices $u, v \in \Omega$, there exists $g \in W$ such that

$$u^g = v.$$

The action of W on ordered pairs preserves every orbital. Therefore, if $(x, y) \in \Delta$, then

$$(x^g, y^g) \in \Delta.$$

Consequently, every element of W induces an automorphism of Γ. Since W is transitive on the vertices, Γ is vertex-transitive.

Now let (u_1, v_1) and (u_2, v_2) be arbitrary arcs of Γ. By definition,

$$(u_1, v_1), (u_2, v_2) \in \Delta.$$

Since Δ is a single orbit under W, there exists $g \in W$ such that

$$(u_1, v_1)^g = (u_2, v_2).$$

Thus

$$u_1^g = u_2 \qquad \text{and} \qquad v_1^g = v_2.$$

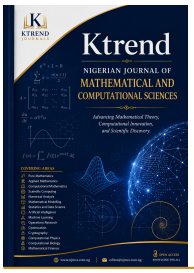
Hence g maps the first arc to the second, proving that Γ is arc-transitive.

(2) The out-degree.
The out-neighbour set of α is

$$\Gamma^+(\alpha) = \{\delta \in \Omega : (\alpha, \delta) \in \Delta\}.$$

We claim that

$$\Gamma^+(\alpha) = \beta^{W_\alpha}.$$



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First, let $\delta \in \beta^{W_\alpha}$. Then $\delta = \beta^g$ for some $g \in W_\alpha$. Since g fixes α,

$$(\alpha, \beta)^g = (\alpha, \beta^g) = (\alpha, \delta) \in \Delta.$$

Therefore, $\delta \in \Gamma^+(\alpha)$.

Conversely, suppose that $\delta \in \Gamma^+(\alpha)$. Then

$$(\alpha, \delta) \in \Delta = (\alpha, \beta)^W.$$

Hence there exists $g \in W$ such that

$$(\alpha, \beta)^g = (\alpha, \delta).$$

It follows that $\alpha^g = \alpha$, so $g \in W_\alpha$, and $\delta = \beta^g$. Thus $\delta \in \beta^{W_\alpha}$.

The claim follows, and consequently

$$d^+(\alpha) = |\Gamma^+(\alpha)| = |\beta^{W_\alpha}|.$$

Because Γ is vertex-transitive, every vertex has the same out-degree. Therefore,

$$\boxed{d^+(\Gamma) = |\beta^{W_\alpha}|.}$$

(3) The in-degree.

The in-neighbour set of α is

$$\Gamma^-(\alpha) = \{\delta \in \Omega : (\delta, \alpha) \in \Delta\}.$$

By the definition of the reverse orbital,

$$(\delta, \alpha) \in \Delta \quad \Longleftrightarrow \quad (\alpha, \delta) \in \Delta^*.$$

Hence

$$\Gamma^-(\alpha) = \{\delta \in \Omega : (\alpha, \delta) \in \Delta^*\}.$$

The set Δ^* is itself an orbital. If $(\alpha, \gamma) \in \Delta^*$, then the same argument used in part (2), applied to Δ^*, gives

$$\Gamma^-(\alpha) = \gamma^{W_\alpha}.$$

Consequently,

$$d^-(\alpha) = |\gamma^{W_\alpha}|.$$

The in-degree is constant because Γ is vertex-transitive. Therefore,

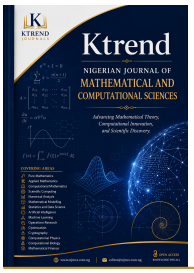
$$\boxed{d^-(\Gamma) = |\gamma^{W_\alpha}|.}$$

(4) The non-self-paired case.

By definition, Δ is self-paired if and only if $\Delta = \Delta^*$. Therefore, if Δ is non-self-paired, it follows directly that

$$\boxed{\Delta^* \neq \Delta.}$$

This completes the proof. □



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2.8. Complete Multipartite Graphs

Definition 2.32 (Complete multipartite graph)**.** Let

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_t$$

be a partition of a finite set Ω into nonempty, pairwise disjoint subsets. A graph Γ with vertex set Ω is called a complete multipartite graph if two distinct vertices are adjacent if and only if they belong to different parts of the partition.

Thus, for $u, v \in \Omega$ with $u \neq v$,

$$u \sim v \quad \Longleftrightarrow \quad u \in \Omega_i,\ v \in \Omega_j \text{ for some } i \neq j.$$

Theorem 2.33 (Complete Multipartite Graph)**.** *Let*

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_t$$

be a partition into nonempty, pairwise disjoint sets, and let Γ *be the graph in which two distinct vertices are adjacent precisely when they belong to different parts. Put*

$$|\Omega_i| = q_i, \qquad i = 1, 2, \ldots, t.$$

Then Γ *is a complete multipartite graph, and*

$$\boxed{|V(\Gamma)| = \sum_{i=1}^{t} q_i.}$$

If $v \in \Omega_i$*, then*

$$\boxed{d(v) = \sum_{\substack{j=1 \\ j \neq i}}^{t} q_j = \left(\sum_{j=1}^{t} q_j\right) - q_i.}$$

If $t \geq 2$*, the graph* Γ *is regular if and only if all the part sizes are equal.*

Proof. We prove the theorem in three steps.

Step 1: The number of vertices.

Since the sets $\Omega_1, \Omega_2, \ldots, \Omega_t$ form a partition of Ω, they are pairwise disjoint and their union is Ω. Therefore, the cardinality of Ω is the sum of the cardinalities of the individual parts:

$$|\Omega| = |\Omega_1| + |\Omega_2| + \cdots + |\Omega_t|.$$

Using $|\Omega_i| = q_i$, we obtain

$$\boxed{|V(\Gamma)| = \sum_{i=1}^{t} q_i.}$$

Step 2: The degree of a vertex.

Let $v \in \Omega_i$. By the definition of a complete multipartite graph, v is adjacent to every vertex that lies outside Ω_i, and it is not adjacent to any other vertex in Ω_i.

Consequently, the neighbourhood of v is

$$N(v) = \bigcup_{\substack{j=1 \\ j \neq i}}^{t} \Omega_j.$$

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The parts are pairwise disjoint, so

$$\begin{aligned} d(v) &= |N(v)| \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{t} |\Omega_j| \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{t} q_j. \end{aligned}$$

Equivalently,

$$\boxed{d(v) = \sum_{j=1}^{t} q_j - q_i.}$$

This establishes the degree formula.

Step 3: Characterisation of regularity.

Assume $t \geq 2$. A graph is regular if all its vertices have the same degree.

Every vertex in Ω_i has degree

$$d_i = \sum_{j=1}^{t} q_j - q_i.$$

Let

$$N = \sum_{j=1}^{t} q_j.$$

Then $d_i = N - q_i$.

Suppose first that Γ is regular. For any $i, k \in \{1, \ldots, t\}$, regularity gives

$$N - q_i = N - q_k.$$

Subtracting N from both sides yields

$$q_i = q_k.$$

Since i and k were arbitrary, all part sizes are equal.

Conversely, suppose that all part sizes are equal:

$$q_1 = q_2 = \cdots = q_t = q.$$

For any vertex $v \in \Omega_i$, the degree formula gives

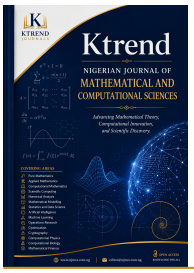
$$d(v) = \sum_{\substack{j=1 \\ j\neq i}}^{t} q = (t-1)q.$$

This value is independent of the choice of v. Hence Γ is regular.

Therefore, for $t \geq 2$,

$$\boxed{\Gamma \text{ is regular} \iff q_1 = q_2 = \cdots = q_t.}$$

The proof is complete. □



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Corollary 2.34. *If a complete multipartite graph has $t \geq 2$ parts, each of size q, then*

$$\boxed{|V(\Gamma)| = tq}$$

and

$$\boxed{d(\Gamma) = (t-1)q.}$$

Its number of edges is

$$\boxed{|E(\Gamma)| = \binom{t}{2} q^2.}$$

Proof. There are t parts, each containing q vertices, so the number of vertices is tq. Each vertex is adjacent to every vertex in the other $t-1$ parts, giving degree $(t-1)q$.

Every edge joins two different parts. There are $\binom{t}{2}$ ways to choose a pair of distinct parts, and each such pair contributes q^2 edges. Thus

$$|E(\Gamma)| = \binom{t}{2} q^2.$$

□

Remark 2.35. The original theorem states the regularity condition as equal part sizes, with a separate reference to $t = 2$. For $t \geq 2$, the equal-part-size condition already includes the case $t = 2$. If $t = 1$, the graph has no edges and is 0-regular regardless of the size of its single part.

3. CONCLUDING REMARKS

This study examined suborbital graphs arising from the product actions of $S_m \wr S_n$, $A_m \wr S_n$, $C_m \wr S_n$, and $D_{2m} \wr S_n$ on X^n. The results established relationships between base-group orbit structures and the classification, pairing properties, and graph-theoretic characteristics of the resulting suborbital graphs. The findings show that symmetric and alternating group actions can produce adjacency structures governed by Hamming distance, while cyclic and dihedral group actions reflect their respective permutation structures. Overall, the study provides a framework for understanding how base-group actions and coordinate permutations influence suborbital graphs.

Data Availability Statement

No new data were created or analysed in this study.

Declarations

The authors declare that there are no conflicts of interest or competing interests regarding the publication of this paper.

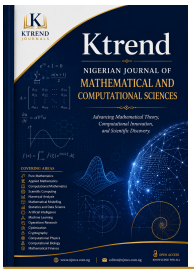
Author Contributions

Salihu Lawan Aliyu: Conceptualization, methodology, formal analysis, investigation, and writing—original draft.

Shuaibu Garba Ngulde:Supervision, validation, and formal analysis.

Bukar Goni Ahmadu:Supervision, Software, data curation, visualization, and validation.

Babagana Alhaji Madu: Supervision, resources, project administration, and writing, review and editing.



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All authors contributed to the development of the research, discussed the results, reviewed the manuscript, and approved the final version for publication.

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